

UNIVERSIDAD DE MURCIA

Escuela de Doctorado

TESIS DOCTORAL

Cantidades entrópicas para el estudio de simetrías generalizadas. Simetrías no invertibles y anomalías.

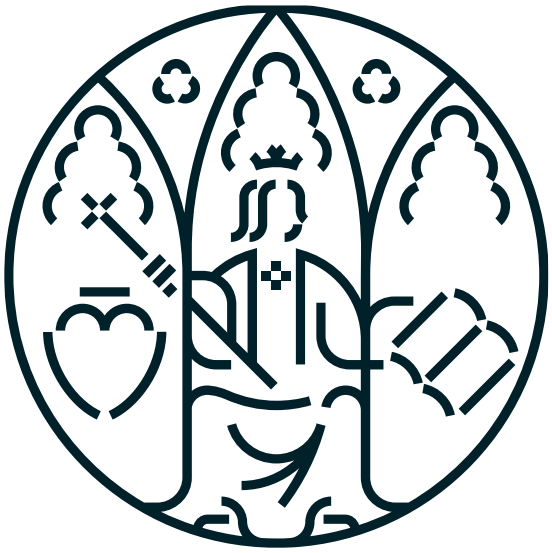
Entropic quantities for the study of generalized symmetries. Non-invertible symmetries and anomalies.

AUTOR/A Pablo Saura Bastida

DIRECTOR/ES Javier Molina Vilaplana

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Entropic quantities for the study of generalized symmetries. Non-invertible symmetries and anomalies./Cantidades entrópicas para el estudio de simetrías generalizadas. Simetrías no invertibles y anomalías.

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Agradecimientos

First and foremost I would like to thank my supervisors Javier and Karl for having taught me all I know of anomalies, generalized symmetries and theoretical physics in general. From them I have not only learned physics, but also how to be a good scientist and do research. I am really grateful for your support during all these years, for being accessible for any help I needed and for all the time you have spent with me.
Secondly I am thankful for my collaborators and the people I have had the pleasure to share discussions: Germán Sierra, Pau García Romeu, David García Fariña, Sergio Morales Tejera, Iñaki García Etxebarria, Nabil Iqbal, Arpit Das and Finn Gagliano. Thank you for all I have learnt from you.
I am very grateful to Fundación Séneca for supporting my career and granting me the fellowship 21609/FPI/21. Fundación Séneca. Región de Murcia (Spain).
Now, on a more personal note I would like to thank all the people that have been by my side all these years, giving me their support and love, and ultimately have made possible for me to sail through this period. For that I would like to do it in Spanish.
Quiero agradecer a todos mis amigos de Faro moral: Pau, Dani, David, mi hermanita Marina, Fer, Nacho, Lorenzo, Edu, Matilda, Luca, Gonzalo y Michelangelo. Por haberme acompañado durante este periodo y haber hecho de Madrid y del IFT un lugar en el que me sienta como en casa, por las tardes de PhDrinks y las noches de Chaqueño.
También quiero agradecer a mis compañeras de desquicios Mali, Ada y Emma. Por haber compartido todas nuestras locuras y haberlas hecho un poco menos locas fuera de nuestra cabeza, por todo el apoyo y por vivir siempre las historias mas delirantes.
A mis amigas de Murcia, en especial a Paula por haber estado siempre en los momentos más difíciles, a Elisa y a Marina (Mili), por haberme dado las mayores risas y haberme hecho sentir siempre cómodo expresándome tal como soy.
A Paquito, Juanmi, Isa, Vanessa, Marta y Luis, por nuestras comidas de viejos, rutas y reuniones.
A Ana y a Carlos por los planes alternativos y las noches de Drag Race. Por haberme acompañado comentando todos nuestros lipsyncs favoritos.
A mi grupo de neurodivergentes Antonio, Elena, Pablo y Pedro.
Gracias también a los Patafísicos: Antonio, Eloisa, Joaquín, Pedro, Víctor, Leandro y Alfredo. Por haber compartido desde la carrera el camino, a veces tortuoso,

de la física, y haber estado siempre presentes celebrando exámenes o llorándolos con unos litricos en el campus.
A mis amigos del instituto Nuria, Pablo y Javi.
También quería agradecer a mis compañeros de doctorado: Juanra, Giacomo, Elena y Raúl, por haber hecho un poco más fácil venir a la universidad de Murcia día tras día.
A Sonia por los cafés, las cervezas, las charlas y las risas. A Alexandra por haber hecho las comidas más divertidas y a Nieves por las grandes recomendaciones de podcast.
También quería agradecer a todos los lugares en los que he pasado tanto tiempo durante esta etapa, donde dejo recuerdos y conversaciones inolvidables. Queria agradecer al Bahía Blanca, en especial a Leida, al Ayuyang Karaoke Bar por todas las performance que he podido hacer con mis amigos allí, a la discoteca Rincón Chaqueño, a Maggie de Murcia Secreto por organizar las noches de trivial de los jueves, a Maraña, a la sala Revolver y a la sala REM.
To my friends from Newcastle and Durham Antonio, Sunil and Salman, for having made my stay more fun and keeping me afloat during that period.
A mi prima Berta por darme colágeno.
A Papá, a Mamá, a Jorge, a mis abuelos y al resto de mi familia.
Y, por supuesto, a Gabo.

Publications

This thesis is structured as a compendium of the following publications:

- *Anomalous transport from geometry.*
  Karl Landsteiner, Sergio Morales-Tejera and **Pablo Saura-Bastida**.
  Phys. Rev. D, 107(12):125003, 2023. [arXiv:2212.14088 [hep-th]].

- *Categorical-symmetry resolved entanglement in conformal field theory.*
  P. Saura-Bastida, A. Das, G. Sierra and J. Molina-Vilaplana.
  Phys. Rev. D, 109(10):105026, 2024. [arXiv:2402.06322 [hep-th]].

- *Generalized symmetry resolution of entanglement in conformal field theory for twisted and anyonic sectors.*
  Arpit Das, Javier Molina-Vilaplana and **Pablo Saura-Bastida**.
  Phys. Rev. D, 110(12):125005, 2024. [arXiv:2409.02162 [hep-th]].

- *Entropic Order Parameters for Categorical Symmetries in 2D-CFT.*
  Javier Molina-Vilaplana, **Pablo Saura-Bastida** and Germán Sierra.
  Entropy, 26(12):1064, 2024. [arXiv:2409.13460 [hep-th]].

Other articles published during the development of thesis but not related to the topic, thus not included, are:

- *Pseudospectra of complex momentum modes.*
  David García Fariña, Karl Landsteiner, Pau G. Romeu and **Pablo Saura-Bastida**.
  JHEP 01 (2025), 185. [arXiv:2407.06104 [hep-th]].

- *Pseudospectra of holographic diffusion: gauge fields breaking free from the master scalar.*
  David García Fariña, Karl Landsteiner, Pau G. Romeu and **Pablo Saura-Bastida**.
  Currently accepted for publication in JHEP. [arXiv:2602.02659 [hep-th]].

Resumen en español

Las simetrías son uno de los conceptos más fundamentales en física teórica. Estas definen corrientes y cargas conservadas a través del teorema de Noether, organizan espacios de Hilbert, clasifican operadores según cómo transforman bajo la acción de la simetría y definen diferentes fases de acuerdo con el paradigma de Landau. Las simetrías han sido ampliamente utilizadas en física clásica debido a las ventajas que presentan en el estudio de estas teorías. Sin embargo, al tratar de extender estas simetrías clásicas a teorías cuánticas algunas de ellas pueden no sobrevivir al proceso de cuantización. Cuando una simetría clásica no se mantiene a nivel cuántico decimos que dicha simetría es anómala.

Las anomalías son uno de los fenómenos cuánticos más sorprendentes. El ejemplo más conocido de simetría anómala es la simetría quiral en teoría de fermiones sin masa. La simetría quiral no sobrevive el proceso de cuantización en presencia de un campo gauge o campo gravitacional externos. Esto es conocido como la anomalía quiral. La anomalía quiral rompe las leyes de conservación de la corriente de número de fermiones. Esto produce un desequilibrio entre fermiones levógiros y fermiones dextrógiros, aunque el número total de fermiones sigue siendo conservado. Cuando se considera esta teoría en un medio con temperatura y potencial químico este desequilibrio se manifiesta como corrientes de transporte. Este fenómeno se conoce como efectos de transporte anómalo. Los efectos más relevantes se manifiestan cuando se exhiben estos fermiones a un campo magnético externo: Efecto Magnético Quiral (CME por sus siglas en inglés) o a rotación: Efecto Vortical Quiral (CVE por sus siglas en inglés). Se cree que estos fenómenos juegan un papel muy importante en algunos sistemas descritos por fermiones quirales, como los semimetales de Weyl y el plasma de quarks y gluones.

En los últimos años, nuevas maneras de estudiar simetrías y anomalías han surgido entre la comunidad científica. Transformaciones de simetría pueden ser escritas como operadores topológicos. De manera correspondiente, cada operador topológico en una teoría implementa una simetría. En este lenguaje, las simetrías ordinarias son implementadas por operadores topológicos de co-dimensión uno cuya multiplicación sigue las reglas de composición de un grupo. Sin embargo, no todos los operadores topológicos satisfacen necesariamente estas condiciones. Esto abre las posibilidades para la existencia de nuevas simetrías, conocidas como simetrías generalizadas. Las dos principales formas de simetrías generalizadas son las simetrías *higher-form* y las simetrías no invertibles. Las primeras son simetrías implementadas por operadores topológicos definidos en superficies de

co-dimensión mayor que uno, pero que satisfacen reglas de composición de grupo, mientras que las segundas no satisfacen una multiplicación de grupo, sino una estructura matemática conocida como categoría de fusión. Por este motivo, las simetrías no invertibles también son conocidas como simetrías categóricas.

La adición de las simetrías generalizadas al panorama de la física teórica ha contribuido al entendimiento y unificación de fenómenos físicos que se creían no asociados a simetrías. Algunos importantes ejemplos de esto incluyen el entendimiento del fotón como un bosón de Goldstone asociado a la rotura espontánea de una simetría *higher-form* y la explicación de algunas fases de teorías en principio no descritas por el paradigma de Landau, como fases de teorías gauge y fases topológicas protegidas por la simetría (fases SPT, por sus siglas en inglés). Esto último inicia un nuevo programa de caracterización de fases por simetrías incluyendo simetrías generalizadas, conocido como el paradigma de Landau generalizado. En esta tesis nos centraremos en el estudio y caracterización de simetrías no invertibles en teorías de baja dimensionalidad.

Las simetrías categóricas son implementadas por operadores topológicos que cumplen las reglas de composición de una categoría de fusión. Esto significa que la multiplicación de dos operadores no tiene como resultado un único operador, sino una suma de ellos. Esta estructura es posible en teorías cuánticas debido a la superposición. Las simetrías categóricas se encuentran presentes en muchas de estas teorías, siendo un ejemplo muy importante de ellas la anomalía quiral. En el paradigma de las simetrías generalizadas, la anomalía quiral puede ser escrita en términos de un operador no invertible. A pesar de tener este ejemplo, no se conocen muchas otras simetrías no invertibles en teorías de más de dos dimensiones. Sin embargo, en teorías en dos dimensiones se pueden encontrar muchos ejemplos de simetrías categóricas.

Un caso especial de teorías en las que las simetrías no invertibles se encuentran presentes son las teorías de campos conformes en dos dimensiones. En estas teorías se pueden encontrar construcciones explícitas de simetrías categóricas. Debido a la baja dimensionalidad de estas teorías, solo es posible construir simetrías definidas en superficies de dimensión uno, esto es, en líneas. Estas simetrías reciben el nombre de líneas de defectos topológicos (TDLs), que en general siguen la estructura de una categoría de fusión. Ejemplos de simetrías no invertibles en estas teorías son la dualidad de Kramers-Wannier en el modelo de Ising crítico o la dualidad T en la teoría del bosón compacto.

Uno de los fenómenos más interesantes que diferencian teorías cuánticas de teorías clásicas es el entrelazamiento. El entrelazamiento es responsable de las correlaciones no locales entre dos subsistemas en teorías cuánticas. Este fenómeno se caracteriza por una cantidad de información cuántica: la entropía de entrelazamiento. La entropía de entrelazamiento mide correlaciones no locales entre dos regiones separadas espacialmente. Esta cantidad se define a través

de la matriz densidad reducida, esto es, la matriz densidad que representa los grados de libertad de un estado restringido a un subsistema. Para definirla, es necesario dividir el espacio de Hilbert en las regiones. Sin embargo, en teorías cuánticas de campos esto no es siempre posible ya que el espacio de Hilbert es, en general, no separable. Por tanto, para definir entropía de entrelazamiento, es necesario dar una prescripción para separar el espacio de Hilbert. Una de las prescripciones más comunes consiste en definir dos discos en la separación entre las regiones consideradas, los puntos de entrelazamiento, y definir condiciones de contorno conformes en dichos puntos. En esta prescripción, el radio de los discos actúa como un regulador ultravioleta, siendo la entropía de entrelazamiento una cantidad divergente en el ultravioleta, según el regulador va a cero.

En presencia de una simetría global es posible definir cantidades entrópicas sensibles a la estructura de la simetría. Una de estas cantidades es la entropía de entrelazamiento resuelta por la simetría (SREE). Esta cantidad da cuenta de cómo el entrelazamiento se distribuye entre los sectores de carga asociados a la simetría global. Cómo la entropía de entrelazamiento se distribuye entre los sectores de carga depende del tamaño de estos. En el caso de simetrías ordinarias, esto corresponde al tamaño de las representaciones irreducibles del grupo correspondiente. Una característica interesante de esta cantidad es que, para simetrías abelianas, la entropía de entrelazamiento resuelta por la simetría no depende de la representación irreducible. En otras palabras, todas las representaciones irreducibles tienen el mismo tamaño en este caso. Este resultado se conoce como equipartición de la entropía de entrelazamiento. En términos físicos esto se refiere a que la entropía de entrelazamiento se distribuye por igual entre todos los sectores de carga.

Otras aplicaciones interesantes de las cantidades entrópicas es la detección de fases de la materia. Esto se puede conseguir a través de cantidades entrópicas que permitan comparar entre dos matrices densidad, como la entropía relativa o la asimetría de entrelazamiento. La entropía relativa cuantifica la probabilidad de confundir un estado con otro como resultado en una medida experimental. Para caracterizar fases de la materia utilizando esta cantidad es necesario definir una función de simetrización. Dada una simetría global, dicha función extrae la parte simétrica de una matriz densidad. Si dicha matriz densidad ya es simétrica, permanecerá invariante después de aplicar la función de simetrización. La entropía relativa permite comparar entre una matriz densidad que representa un estado de vacío y su versión simetrizada. Esta cantidad se anula si y solo si el vacío es simétrico, correspondiendo con una fase en la que no hay rotura espontánea de simetría. Por tanto, una entropía relativa no nula es una señal de rotura espontánea de simetría en este contexto, haciendo de esta cantidad una candidata para parámetro de orden entrópico.

El objetivo principal de esta tesis es el estudio de anomalías, fenómenos de

transporte anómalo y simetrías no invertibles. Para ello utilizaremos distintos métodos, desde el estudio de los fenómenos de transporte anómalo desde la teoría cuántica de campos hasta definir cantidades entrópicas para el estudio y caracterización de simetrías no invertibles y fases asociadas a rotura espontánea de estas simetrías. Este texto está estructurado como un compendio de artículos publicados durante el desarrollo de esta tesis.

La primera investigación consiste en el estudio de fenómenos de transporte anómalos, el efecto magnético quiral y el efecto vortical quiral, mediante integración directa de las divergencias anómalas en teoría de fermiones sin masa en cuatro dimensiones. Esto conecta directamente la anomalía en la teoría cuántica con los fenómenos de transporte que exhiben fermiones quirales sujetos a una temperatura y potencial químico finitos. Para establecer esta conexión, es necesario una configuración que interpole entre la teoría cuántica de campos y una distribución termodinámica. Esto se puede conseguir mediante el uso de agujeros negros con ciertas propiedades, que son, carga magnética en el caso del CME y rotación en el caso del CVE. Con estas configuraciones, es posible integrar la anomalía obteniendo los efectos de transporte anómalo en cuatro dimensiones.

El segundo y tercer artículos se centran en el estudio de la entropía de entrelazamiento resuelta por la simetría en teorías de campos conformes en dos dimensiones. Para ello extendemos los conceptos necesarios para el estudio de dicha cantidad para incluir simetrías categóricas. Los sectores de carga asociados a una simetría ordinaria están dados por representaciones irreducibles de un grupo. El análogo para una categoría de fusión son las representaciones irreducibles del álgebra de tubo. Para resolver con respecto a la simetría es necesario insertar projectores asociados a estas representaciones irreducibles en la función de partición de la teoría, definiendo así una matriz densidad proyectada en un sector de carga. Calcular entropía de entrelazamiento requiere separar el espacio de Hilbert en dos regiones. Proyectar en sectores de carga no siempre es compatible con esta separación. Cuando esta incompatibilidad existe decimos que la simetría es anómala, ya que no se puede definir a nivel local.

Las anomalías nos impiden resolver con respecto a la simetría. En nuestra formulación, las anomalías se manifiestan explícitamente como la no existencia de condiciones de contorno simétricas que nos permitan dividir el espacio de Hilbert en dos partes. Por tanto, el estudio de anomalías a este nivel es importante para nuestra investigación. Si la simetría no es anómala, es posible calcular la resolución de la entropía de entrelazamiento. Para simetrías categóricas encontramos una característica muy interesante. Incluso si la categoría de fusión es abeliana, la equipartición de entrelazamiento se rompe debido a la dimensión cuántica no trivial de los operadores que implementan la simetría no invertible.

Finalmente, en el último artículo se trata el estudio de parámetros de orden entrópicos para simetrías no invertibles. Para ello, proponemos una función de

simetrización que incluye simetrías categóricas y calculamos la entropía relativa entre estados de vacío y su versión simetrizada en teorías de campos conformes en dos dimensiones con simetrías no invertibles. Aplicamos este cálculo a ejemplos conocidos de simetrías categóricas, como la simetría de Fibonacci, obteniendo resultados numéricos que reproducen valores no nulos en casos en que la simetría es categórica. Este trabajo presenta una primera aproximación a la introducción de simetrías no invertibles en el programa de los parámetros de orden entrópicos en el contexto del paradigma de Landau generalizado.

A lo largo de este texto presentaremos las publicaciones mencionadas anteriormente, contextualizándolas en el marco teórico del estudio de simetrías generalizadas, en particular simetrías no invertibles, anomalías en teorías cuánticas de campos y cantidades entrópicas como herramientas para la caracterización de dichas simetrías. Presentando los resultados principales que han sido encontrados en las publicaciones y perspectivas futuras de estas investigaciones.

Abstract

Symmetries are one of the most fundamental principles of theoretical physics. They define conserved currents and charges through Noether's theorem, organize the Hilbert space, classify operators based on how they transform under the symmetry and define different phases according to the Landau paradigm. Symmetries have been widely used in classical physics due to the advantages they present in the study of those theories. However, when trying to extend classical symmetries to quantum theories, some of them may not survive the process of quantization. When a classical symmetry does not hold at quantum level we say the symmetry is anomalous.

Anomalies are one of the most shocking quantum phenomena. The best known example of anomalous symmetry is the chiral symmetry in massless fermion theory. The chiral symmetry does not survive the process of quantization in presence of an external gauge or gravitational field. This is known as the chiral anomaly. The chiral anomaly spoils the conservation laws associated to the chiral fermion number current. This produces an imbalance between left- and right-handed fermions, though the total number is still conserved. When one considers such theory in a medium with temperature and chemical potential this imbalance manifests as transport currents. This phenomenon is known as anomalous transport effects. The most important effects manifest when exhibiting the chiral fermions to an external magnetic field: Chiral Magnetic Effect (CME) or to rotation: Chiral Vortical Effect (CVE). These phenomena are believed to play a very important role in some systems described by chiral fermions, such as Weyl semimetals and the quark-gluon plasma.

In the recent years, new approaches to study symmetries and anomalies have caught the attention of the community. Symmetry transformations can be written as topological operators. Correspondingly, every topological operator will implement a symmetry in the theory. In this language, ordinary symmetries are implemented by co-dimension one topological operators that multiply following the composition rules of a group. However, topological operators do not necessarily satisfy those conditions. This opens the possibilities for new symmetries to exist, which are the so-called generalized symmetries. There are two main generalizations: higher form symmetries and non-invertible symmetries. The former ones are symmetries implemented by topological operators supported in manifolds of co-dimension greater than one, but satisfying group-like fusion rules, while the latter ones do not fulfill group multiplication, but instead follow the mathemati-

cal structure of a fusion category. For this reason, non-invertible symmetries are also referred to as categorical symmetries.

The addition of generalized symmetries to the framework has helped in the understanding and unification of physical phenomena that were thought to lie outside the symmetry paradigm. Notorious examples include the understanding of the photon as a Goldstone mode associated to spontaneous symmetry breaking of a higher-form symmetry and the explanation of some phases of theories that were thought laying outside the Landau paradigm, such as phases of gauge theories and symmetry protected topological (SPT) phases. This opened a new program of characterizing phases by symmetries including generalized symmetries, called the generalized Landau paradigm. This thesis is focused in the study and characterization of non-invertible symmetries in lower dimensionality theories.

Categorical symmetries are implemented by topological operators following a fusion category composition rules. This means that multiplying two operators does not have one single operator as a result but a sum of them. This structure is possible in quantum theories due to superposition. Categorical symmetries are ubiquitous in this theories, being a very important example of them the chiral anomaly. In the generalized symmetry paradigm, the chiral anomaly can be recasted as a non-invertible operator. Despite having this example, there are not many others studied in $d > 2$ dimensions. However, in $2-$dimensional theories we can find a lot of physical examples of non-invertible symmetries.

One special subset of theories in which one can find categorical symmetries are $2-$dimensional conformal field theories. In those theories one can find explicit constructions of categorical symmetries. Due to the low dimensionality of the theories, it is only possible to define symmetries supported in 1-dimensional manifolds, that is, in lines. These symmetries receive the name of Topological Defect Lines (TDLs), which in general follow a categorical structure. Examples of non-invertible lines in this theories are the Kramers-Wannier duality of the critical Ising model or $T-$duality of the compact boson.

One of the most interesting phenomena that differentiates quantum theories from classical ones is that of entanglement. Entanglement accounts for non-local quantum correlations between two subsystems. This phenomenon is characterized by a quantum information quantity: the entanglement entropy. Entanglement entropy measures non-local correlations between two space separated regions. This quantity is defined through the reduced density matrix, that is, the density matrix representing the degrees of freedom of a state supported on a subsystem. In order to define it one needs to split the Hilbert space into the regions. In quantum field theories this is not always possible as Hilbert spaces are, in general, not decomposable. Thus, to define entanglement entropy, one has to give a prescription on how to split a Hilbert space. One of the most common prescriptions is to

define two disks at the separation between the regions, the entangling points, and define some conformal boundary conditions there. In this prescription, the radius of the disks acts a UV regulator, making entanglement entropy a UV divergent quantity as the regulator goes to zero.

In presence of a global symmetry, it is possible to define entropic quantities that probe the structure of the symmetry. One of this quantities is Symmetry Resolved Entanglement Entropy (SREE). This quantity measures how entanglement is distributed along charged sectors associated to the global symmetry. How the entanglement entropy is distributed in charged sectors depends on the size of such sectors. For ordinary symmetries this corresponds to sizes of irreducible representations of the corresponding group. An interesting feature is that, for abelian groups, symmetry resolved entanglement entropy does not depend on the irrep. In other words, all irreps are of the same size in that case. This result is known as equipartition of entanglement entropy which, in physical terms, refers to entanglement entropy distributing equally along all charged sectors.

Other interesting application of entropic quantities is to detect phases of matter. This can be achieved by using entropic quantities that compare two density matrices, such as relative entropy or entanglement asymmetry. Relative entropy quantifies the probability of confusing one state with the other in a experimental setup. In order to characterize phases of matter using this quantity a symmetrizer map is needed. Given a global symmetry, such map projects out the non-symmetric part of a density matrix. If the density matrix is already symmetric it will remain invariant under symmetrization. Relative entropy allows to compare between a density matrix representing a vacuum state and its symmetrized version. This quantity will be zero if and only if the vacuum is symmetric, corresponding to a phase where there is not spontaneous symmetry breaking. Therefore, a non-zero relative entropy is a signature of spontaneous symmetry breaking, making this quantity an entropic order parameter.

The main objective of this thesis is the study of anomalies, anomalous transport phenomena and non-invertible symmetries. For that we use different approaches, from studying anomalous transport effects from the quantum field theory to defining entropic quantities for the study and characterization of non-invertible symmetries and phases associated to spontaneous breaking of them. The text is structured as a compendium of the papers published during the development of the thesis.

The first research consists on the study of anomalous transport effects, the CME and the CVE, via direct integration of the anomalous divergencies in massless fermion theory in 4-dimensions. This connects directly the anomaly in the quantum theory with the transport phenomena exhibited by chiral fermions at finite temperature and chemical potential. In order to establish this connection we need a background that interpolates between the QFT and a thermal ensemble.

That can be achieved by black holes with certain properties, those are, magnetic charge for the CME and rotation in the case of the CVE. With these backgrounds, it is possible to integrate the anomaly and derive the anomalous transport effects in 4-dimensions.

The second and third papers focus on the study of symmetry resolved entanglement entropy in 2-dimensional conformal field theories. For that we extend the necessary concepts for SREE to include categorical symmetries. Charged sectors associated to ordinary group-like symmetries are given by irreducible representations of the group. The analogue for a category are the irreducible representations of the tube algebra. In order to symmetry resolve we need to insert the projectors associated to these irreps into the partition function of the theory, thus defining a projected density matrix into a charge sector. Computing entanglement entropy requires splitting the Hilbert space in two regions. Projecting into charged sectors is not always compatible with this separation. When this incompatibility exists we say the symmetry is anomalous, as it fails to hold locally.

Anomalies prevent us to symmetry resolve. In our approach they manifest explicitly as the non-existence of boundary symmetric conditions allowing us to split the Hilbert space in two pieces. Therefore, the study of anomalies at this level is of importance to us. If the symmetry is non-anomalous, we may compute symmetry resolved entanglement entropy. For categorical symmetries a very interesting feature of this quantity is found. Even if the fusion category is abelian, entanglement equipartition is broken due to the non-trivial quantum dimensions of the operators implementing the non-invertible symmetry.

Finally, the last paper addresses the study of entropic order parameters for non-invertible symmetries. For that, we propose a symmetrization map that includes categorical symmetries and compute the relative entropy between vacuum states and their symmetrized version in 2-dimensional conformal field theories with non-invertible symmetries. We apply this computation to known examples categorical symmetries, like the Fibonacci symmetry, obtaining numerical results exhibiting non-zero values when the symmetry is categorical. This work amounts for the first approach to include non-invertible symmetries in the program of entropic order parameters in the context of the generalized Landau paradigm.

Through this text we will present the aforementioned publications, contextualizing them in the theoretical framework of the study of generalized symmetries, in particular non-invertible, anomalies in quantum field theories and entropic quantities as a tool for the characterization of those symmetries. Presenting the main results found in the publications and future prospects of this investigations.

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Chapter 1

Introduction

1.1 | Symmetries in Quantum Field Theory

Quantum Field Theory (QFT) is one of the most important theoretical techniques used in modern physics. It is able to successfully describe three of the four interactions found in nature at a quantum and relativistic level. In these theories, symmetries play a fundamental role simplifying them and leading to powerful theorems. Classically, let us consider a field theory described by a Lagrangian density:

$$\mathcal{L}(\phi_a, \partial_\mu \phi_a)\,, \tag{1.1}$$

where $\{\phi_a\}$ is the field content of the theory. We say that an infinitesimal transformation $\delta_\alpha \phi_a = \alpha X_a(\phi)$ is a symmetry if it changes the Lagrangian just by a total derivative:

$$\delta_\alpha \mathcal{L}(\phi_a, \partial_\mu \phi_a) = \alpha \partial_\mu f^\mu\,. \tag{1.2}$$

By varying explicitly the Lagrangian, one can find that this condition is equivalent to:

$$\delta_\alpha \mathcal{L}(\phi_a, \partial_\mu \phi_a) = \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) \right) = \alpha \partial_\mu f^\mu\,, \tag{1.3}$$

after having used the equations of motion. From this expression one can see that it is possible to define a quantity

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) - f^\mu\,, \tag{1.4}$$

such that the conservation equation $\partial_\mu j^\mu = 0$ holds. This result is known as the Noether theorem, and it implies that if there is a continuous symmetry in a theory, a conserved current (or Noether's current) exists.

Noether's theorem holds also at quantum level, where the analogue to the conservation equation of Noether's current are the so-called Ward-Takahashi identities [1]. These identities tell us how Noether currents behave in the presence of charged operators. We define a charged operator as those that transform under the symmetry:

$$\delta \mathcal{O}_q(x) = i\alpha q \mathcal{O}_q(x)\,, \tag{1.5}$$

with q the charge. In a quantum theory, the correlations can be defined in the path integral formalism as

$$\langle \mathcal{O}_q(x)...\rangle = \frac{1}{Z}\int [\mathcal{D}\phi_a]\mathcal{O}_q(x)...\, e^{iS}\,, \tag{1.6}$$

where ... represent arbitrary insertions of operators. If a transformation is a symmetry, then every correlation must be invariant under it:

$$\delta_\alpha \langle \mathcal{O}_q(x)...\rangle = 0\,, \tag{1.7}$$

By computing the variation explicitly

$$\delta_\alpha \langle \mathcal{O}_q(x)...\rangle = \frac{1}{Z}\int [\mathcal{D}\phi_a]\left[\left(\delta_\alpha \mathcal{O}_q(x) - i\alpha \mathcal{O}_q(x)\int d^d y \partial^\mu j_\mu(y)\right)... + \mathcal{O}_q(x)\delta_\alpha(...)\right] e^{iS}\,. \tag{1.8}$$

This must vanish for arbitrary insertions (...), which in particular means that the combination

$$\int [\mathcal{D}\phi_a] i\alpha \left(q\mathcal{O}_q(x) - \mathcal{O}_q(x)\int d^d y \partial^\mu j_\mu(y)\right)... e^{iS}\,. \tag{1.9}$$

This leads to the expression of conservation in quantum theories, also known as the Ward-Takahashi identity:

$$\langle \partial_\mu j^\mu(x)\mathcal{O}_q(y)...\rangle = q\delta^{(d)}(x-y)\langle \mathcal{O}_q(x)...\rangle\,. \tag{1.10}$$

At the quantum level, conservation laws are understood locally. The divergence of a conserved current exhibit a non-zero expectation value in the presence of a charged operator located entirely where this operator is inserted. Thus, conservation laws can be violated by contact terms.

In the derivation of the Ward-Takahashi identities we have omitted a very important detail. We have assumed that the measure of the path integral is invariant under the symmetry transformation. However, this is not always the case. Some classical symmetries symmetries do not hold at quantum level. The partition function is always invariant under a renaming of the fields. In particular, after performing a symmetry transformation:

$$\delta_\alpha Z = \delta_\alpha \int [\mathcal{D}\phi_a] \exp(iS[\phi_a]) = 0\,, \tag{1.11}$$

where δ_α here really means the difference between the partition function written in the transformed fields and the original ones. As the transformation is a classical symmetry, the action is invariant under it independently. However, for the

symmetry to hold at a quantum level the measure must also be invariant:

$$\delta_\alpha[\mathcal{D}\phi_a] = 0\,. \tag{1.12}$$

If this does not hold, the classical symmetry is not a symmetry of the quantum theory. In that case, we say that the symmetry is anomalous [2, 3, 4]. Anomalies are ubiquitous in quantum theories, where they play a fundamental role in the description of some physical effects. One of the main objectives of this thesis is the study of anomalous symmetries. In the rest of this section we will introduce one of our cases of interest: the chiral anomaly and its associated physical phenomena.

1.1.1. The chiral anomaly

As said before, in quantum field theory a symmetry might not survive the process of quantization. When this happens we say that the symmetry is anomalous. One of the best studied examples of anomalies in quantum field theories is that of the chiral anomaly of free fermion theory. Let us consider the following theory of free fermions in 4 dimensions:

$$S = \int d^4x \bar{\psi} \not{\partial} \psi\,. \tag{1.13}$$

Classically, this theory has two global $U(1)$ symmetries: the vector symmetry and the axial symmetry. The first one is given by the transformation:

$$\begin{aligned} \psi'(x) &= e^{i\alpha}\psi(x)\,, \\ \bar{\psi}'(x) &= \bar{\psi}(x)e^{-i\alpha}\,, \end{aligned} \tag{1.14}$$

while the latter

$$\begin{aligned} \psi'(x) &= e^{i\alpha\gamma_5}\psi(x)\,, \\ \bar{\psi}'(x) &= \bar{\psi}(x)e^{i\alpha\gamma_5}\,. \end{aligned} \tag{1.15}$$

These transformations have their associated Noether currents, which can be written as:

$$\begin{aligned} j_V^\mu(x) &= \bar{\psi}(x)\gamma^\mu\psi(x)\,, \\ j_A^\mu(x) &= \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\,. \end{aligned} \tag{1.16}$$

The theory can be quantized, we may write the partition function using the path integral formalism

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int d^4x \bar{\psi} \not{D} \psi\right]\,, \tag{1.17}$$

where we have coupled the massless fermions to an external non-dynamical gauge field, in the way:

$$\not{D} = \gamma^\mu(\partial_\mu - iA_\mu\gamma^5)\,. \tag{1.18}$$

Notice that here the gauge field A_μ is not the usual $U(1)$ gauge field introduced in electrodynamics. We have introduced this gauge field alongside a γ_5 in the covariant derivative. This field is called the axial gauge field and plays the role of being a source for the axial current. That means, from the quantum effective action $\Gamma[A]$ defined as

$$e^{i\Gamma[A]} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS[A]}\,, \tag{1.19}$$

the axial current is defined as

$$j^\mu_A = \frac{\delta\Gamma[A]}{\delta A_\mu}\,. \tag{1.20}$$

As we will see later, the path integral (1.17) is no longer invariant under chiral transformations (1.15). While the action is invariant as it is a classical symmetry of the theory, the measure of the path integral gives a non-trivial Jacobian under the chiral transformation [5]. This term spoils the conservation of the axial current. This way of computing the anomaly from the path integral is known as the Fujikawa method. It requires a regularization for the non-trivial Jacobian, as it is, in principle, divergent. Depending on the way of regularizing one may obtain different forms of the non-conserved currents. Parallel to the Fujikawa method one can compute the anomaly using perturbation theory. In that case, there is only one diagram that contributes to the computation, which is the so-called triangle diagram in four dimensions, from where this anomaly sometimes is referred as triangle anomaly. In general, anomalies are protected by a non-renormalization theorem [6], thus they are exact at 1-loop order. Nonetheless, the triangle diagram needs to be regularized in the same way as the Jacobian in the Fujikawa method, being both methods equivalent and giving the different forms of the anomaly based on the regulator chosen. For the purposes of this thesis we will not review both methods in detail. Instead we are going to illustrate the anomaly reviewing the Fujikawa method, for a review on the triangle anomaly one can check [7].

In order to explicitly compute the variation of the measure of the path integral in this theory, let us consider the set of eigenspinors of the Dirac operator, those are the functions $\phi_k(x)$ satisfying:

$$\not{D}\phi_k = \lambda_k\phi_k\,. \tag{1.21}$$

We can now expand general spinors in the basis of these eigenspinors:

$$\begin{aligned}\psi(x) &= \sum_k a_k \phi_k(x)\,, \\ \bar{\psi}(x) &= \sum_k \bar{b}_k \phi_k^\dagger(x)\,,\end{aligned} \tag{1.22}$$

where a_k and $\bar{b}_k$ are Grassmann variables. The eigenspinors are chosen to be orthonormal, thus

$$\int d^4x \bar{\phi}_k \phi_l = \delta_{kl}\,. \tag{1.23}$$

In this expansion, the measure of the path integral can be rewritten as

$$\begin{aligned}\mathcal{D}\bar{\psi}\mathcal{D}\psi &= \left(\det \int d^4x \phi_k^\dagger(x)\phi_l(x)\right)^{-1} \lim_{N\to\infty}\prod_k^N d\bar{b}_k da_k = (\det \delta_{kl})^{-1} \lim_{N\to\infty}\prod_k^N d\bar{b}_k da_k \\ &= \lim_{N\to\infty}\prod_k^N d\bar{b}_k da_k\,.\end{aligned} \tag{1.24}$$

This expansion is useful to compute the variation of the measure under chiral transformations (1.15). An infinitesimal coordinate-dependent chiral transformation in the expansion (1.22) takes the form:

$$\begin{aligned}\psi'(x) &= \sum_k a'_k \phi_k(x) = \psi(x) + i\alpha(x)\gamma_5\psi(x) = \sum_k a_k\phi_k(x) + i\alpha(x)\gamma_5 \sum_k a_k\phi_k(x)\,, \\ \bar{\psi}'(x) &= \sum_k \bar{b}'_k \phi_k^\dagger(x) = \bar{\psi}(x) + i\alpha(x)\bar{\psi}(x)\gamma_5 = \sum_k \bar{b}_k\phi_k(x) + i\alpha(x)\sum_k \bar{b}_k\phi_k(x)\gamma_5\,,\end{aligned} \tag{1.25}$$

from where we read

$$\begin{aligned}a'_k &= a_k + i\sum_l \int d^4x \phi_k^\dagger(x)\alpha(x)\gamma_5\phi_l(x)a_l\,, \\ \bar{b}'_k &= \bar{b}_k + i\sum_l \int d^4x \bar{b}_l\phi_l^\dagger(x)\alpha(x)\gamma_5\phi_k(x)\,.\end{aligned} \tag{1.26}$$

Finally, we can give an explicit expression of the variation of the measure (1.24) under chiral transformations:

$$\prod_k^N d\bar{b}'_k da'_k = \det\left(\delta_{kl} + i\int d^4x \phi_k^\dagger(x)\alpha(x)\gamma_5\phi_l(x)\right)^{-2}\prod_k^N d\bar{b}_k da_k = \mathcal{J}\prod_k^N d\bar{b}_k da_k\,. \tag{1.27}$$

In general, the Jacobian $\mathcal{J}$ is non-trivial, thus the partition function of this theory is not invariant under chiral transformations although the classical action is. The

classical chiral symmetry does not hold at the quantum level, thus we say this symmetry is anomalous.

The non-trivial Jacobian arising from transforming the path integral under axial transformations is, in principle, a divergent term. To give a finite expression for the anomalous current we need to regularize the Jacobian. The choice of regulator is going to determine the final expression we will arrive. Therefore, the anomaly can be presented in different ways depending on the choice of regulator, these are known as the forms of the anomaly. There are two main forms to present the anomaly: the consistent anomaly and the covariant anomaly [8]. The first one is defined as the one satisfying the Wess-Zumino consistency condition. To introduce that, we may work with the quantum effective action defined in (1.19). In presence of the anomaly, the quantum effective action is not invariant under a gauge transformation. We define the anomaly in this language as:

$$\delta_\alpha \Gamma[A] = \mathcal{A}_\alpha \,. \tag{1.28}$$

Gauge transformations must satisfy the gauge algebra. In our case that siumply means that they must commute:

$$[\delta_\alpha, \delta_\beta] = 0 \,, \tag{1.29}$$

which, for the anomaly, implies:

$$\delta_\alpha \mathcal{A}_\beta - \delta_\beta \mathcal{A}_\alpha = 0 \,. \tag{1.30}$$

This is the Wess-Zumino consistency condition. The anomaly satisfying it is called the consistent anomaly. By construction, the consistent anomalous axial current is defined as the functional derivative (1.20), which gives the following (non-)conservation relations:

$$\begin{aligned} \partial_\mu j^\mu_{cons,V}(x) &= 0 \,, \\ \partial_\mu j^\mu_{cons,A}(x) &= \frac{1}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \,. \end{aligned} \tag{1.31}$$

The non-conservation of the axial charge can be understood as an imbalance between left and right-handed chiral fermions. Following this logic, one may express the chiral anomaly in a different way, writing the conservation number currents associated to single-handed fermions. The relation between those currents and the axial-vector symmetries is just

$$\begin{aligned} j^\mu_L &= \frac{1}{2} \left(j^\mu_V + j^\mu_A \right) \,, \\ j^\mu_R &= \frac{1}{2} \left(j^\mu_V - j^\mu_A \right) \,. \end{aligned} \tag{1.32}$$

Thus, the consistent currents read:

$$\begin{aligned} \partial_\mu j^\mu_{cons,L}(x) &= \frac{1}{96\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \\ \partial_\mu j^\mu_{cons,R}(x) &= \frac{-1}{96\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,. \end{aligned} \tag{1.33}$$

In this way we may interpret the anomaly as a non-conservation of single-handed fermions, that is, the number of left- and right-handed fermions is not conserved but the sum is, so the total number of particles is still conserved.

We may notice that the consistent currents, defined as the functional derivative of the quantum effective action, are not gauge invariant. For the single-handed fermion currents:

$$\delta_\alpha j^\mu_{cons;\,L,R} = -\frac{\pm 1}{24\pi^2}\epsilon^{\mu\nu\sigma\rho}\partial_\nu\alpha F_{\sigma\rho} \neq 0\,. \tag{1.34}$$

In views of this, we may be interested in defining a new current by adding a Chern-Simons current such that the non-gauge invariance is canceled. This is the definition of the covariant anomaly:

$$j^\mu_{cov;\,L,R} = j^\mu_{cons;\,L,R} \pm \frac{1}{24\pi^2}\epsilon^{\mu\nu\sigma\rho}A_\nu F_{\sigma\rho}\,. \tag{1.35}$$

The covariant currents read:

$$\begin{aligned} \partial_\mu j^\mu_{cov,L}(x) &= \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \\ \partial_\mu j^\mu_{cov,R}(x) &= \frac{-1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \end{aligned} \tag{1.36}$$

which differ from the consistent ones by a factor of 3 [1]. Finally, these currents can be written in axial-vector form:

$$\begin{aligned} \partial_\mu j^\mu_{cov,V}(x) &= 0\,, \\ \partial_\mu j^\mu_{cov,A}(x) &= \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,. \end{aligned} \tag{1.37}$$

This way of presenting the anomaly is the covariant anomaly in the presence of an external axial gauge field. However, the anomaly also couples to gravity. If we consider the theory of chiral fermions (1.17) in a curved background and in the presence of N $U(1)$ gauge fields, the general form of the currents determining

[1] This factor agrees with the way of regularizing the triangle diagram, as one may choose how to distribute the anomaly among the external legs of the diagram to give a particular regularization. For more details see [7].

the covariant anomaly is:

$$\begin{aligned}\nabla_\mu J^\mu_a =& -\frac{d_{abc}}{32\pi^2}\epsilon^{\mu\nu\sigma\rho}F^b_{\mu\nu}F^c_{\sigma\rho} - \frac{b_a}{768\pi^2}\epsilon^{\mu\nu\sigma\rho}R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\sigma\rho}\,,\\ \nabla_\mu T^{\mu\nu} =& F^{a,\nu\mu}J_{a,\mu} - \frac{b_a}{384\pi^2}\epsilon^{\sigma\rho\alpha\beta}\nabla_\mu\left(F^a_{\sigma\rho}R^{\nu\mu}{}_{\alpha\beta}\right)\,,\end{aligned} \tag{1.38}$$

where the latin subindices $\{a\}_1^N$ are the ones labeling the different gauge fields. The coefficients d_{abc} and b_a appearing in the expressions are the so-called anomaly coefficients. These depend on the difference of charges between left and right-handed fermions: $d_{abc} = \sum_r q^r_a q^r_b q^r_c - \sum_l q^l_a q^l_b q^l_c$ and $b_a = \sum_r q^r_a - \sum_l q^l_a$. The covariant anomaly is completely determined by them.

The anomalous currents play a very important role in quantum many body systems with chiral fermions, where induced transport phenomena arises as a result of the chiral anomaly. These are the so-called anomalous transport effects.

1.1.2. Anomalous transport

Chiral fermions exhibit non-vanishing currents when put on a medium at finite chemical potential or finite temperature and exposed to an external magnetic field or to rotation [9, 10, 11, 12]. These phenomena is exactly characterized by the anomaly coefficients d_{abc} and b_a. This is the so-called anomaly induced transport phenomena: chiral magnetic effect (CME) [13] and chiral vortical effect (CVE) [14], and they are represented by the currents:

$$\begin{aligned}\vec{j}_a =& \frac{d_{abc}}{4\pi^2}\mu_b\vec{B}_c + \left(\frac{d_{abc}}{4\pi^2}\mu_b\mu_c + \frac{b_a}{12}T^2\right)\vec{\Omega}\,,\\ \vec{j}_\epsilon =& \left(\frac{d_{abc}}{8\pi^2}\mu_b\mu_c + \frac{b_a}{24}T^2\right)\vec{B}_a + \left(\frac{d_{abc}}{8\pi^2}\mu_a\mu_b\mu_c + \frac{b_a}{24}\mu_a T^2\right)\vec{\Omega}\,,\end{aligned} \tag{1.39}$$

where $\vec{j}_\epsilon$ is the energy current, $\vec{j}_a$ is the current coupling to the gauge field A^μ_a, that produces the magnetic field $\vec{B}_a$ and $\vec{\Omega}$ is the angular velocity.

The CME and CVE can be realized in quantum many body systems described by chiral fermions exposed by an external magnetic field or rotation. In high energy physics these phenomena are important to describe the quark-gluon plasma [15, 16] produced in heavy ion collisions. These effects are also of importance in condensed matter, where they play a role in systems described by chiral fermions, as Weyl semimetals [17, 18].

The derivation of these currents using statistical physics techniques make their connection to anomalies quite obscure. In order to establish a connection one may try to relate the anomalous divergences (1.38) to the transport phenomena by using techniques that allow to extract thermodynamical behaviors from quantum ensembles. One way of doing this is hydrodynamics.

Hydrodynamics consist on writing every current in a theory in a derivative expansion in terms of the hydrodynamical variables, those are the four velocity u^μ, the temperature T and all chemical potentials associated to internal symmetries μ^a. This gives a total of $d+N$ variables. All possible terms one can write in this expansion are restricted by the symmetries of the theory. This implies they must respect the (non-)conservation equations (1.38), which is a system of $d+N$ equations. In the case of the chiral anomaly, having parity-odd terms in (1.38) allows for axial vectors to appear in the expansion. The only axial vectors available are magnetic fields $\vec{B}_a$ and rotation $\vec{\Omega}$. Thus, to first order in derivatives, the hydrodynamic expansion gives:

$$\begin{aligned} J^\mu_a =& \rho_a u^\mu + \sigma_{B,ab} B^\mu_b + \sigma_{\Omega,a}\Omega^\mu \,, \\ T^{\mu\nu} =& (p+\epsilon)u^\mu u^\nu + p g^{\mu\nu} + \xi_{B,a}(B^\mu_a u^\nu + u^\mu B^\nu_a) + \xi_\Omega(\Omega^\mu u^\nu + u^\mu \Omega^\nu)\,, \end{aligned} \tag{1.40}$$

The 0th derivative order in this expansion corresponds to the ideal fluid, where, ϵ, p and ρ_a are the energy, pressure and particle densities. At first order we have just consider terms proportional to pseudo-vectors, those are the anomalous terms. Generically, we have written these terms precede by some coefficients $\xi_{B,a}$, ξ_Ω, $\sigma_{B,ab}$ and $\sigma_{\Omega,a}$, which are functions of the hydrodynamic variables. These coefficients are totally determined after imposing the second principle of thermodynamics, up to an integration constant preceding the thermal terms of the currents [12, 19]. After determining them in this way, one can reach the transport phenomena (1.39), connecting in this way the CME and CVE with the anomaly in the chiral fermion quantum field theory.

Another way to approach this problem is trying to solve directly the system of equations (1.38), that is, by integrating the divergencies of anomalous currents. For that, it is necessary to choose a background that interpolates between the QFT and a thermal ensemble with a temperature and chemical potential. Preliminary advances in this were made in deriving the CME in 1+1 dimensions and the thermal CVE in 1+3 dimensions [10]. These advances are promising indicating there is a direct connection between the chiral anomaly in QFT and the anomalous transport effects, however, they do not fully characterize the transport currents. As we will see in Chapter 2, one of the purposes of the thesis is to extend this work to derive the full anomalous currents in 1+3 dimensions, establishing a connection between the anomaly in the QFT and both the CME and CVE. This way of obtaining the anomalous currents presents a different derivation to that using hydrodynamics. Although they are both consistent with the results obtained, direct integration allows to obtain all terms of the anomalous currents from the method, including the thermal terms. This solves the ambiguity of those terms in hydrodynamics, where they are introduced as integration constants, situating direct integration as a more general procedure.

Recently, modern ways of studying symmetries have been catching the attention of the community. Powerful techniques based on topology have expanded the way of understanding symmetries and related phenomena, including the chiral anomaly. In the following section we will review this new paradigm, making connection with the other objectives of this thesis: the study of the so-called non-invertible symmetries in quantum theories.

1.2 | Generalized symmetries

In the latest years there has been a shift of paradigm regarding symmetries in quantum field theories. It was realized that the study of the symmetries of a theory is equivalent to the study of the topological sector of said theory [20], being symmetries implemented by topological operators. To illustrate this concept let us use the theory of free complex bosons, described by the action:

$$S = \int d^d x (\partial_\mu \phi^* \partial^\mu \phi + m^2 |\phi|^2) \,. \tag{1.41}$$

This action is invariant under the transformation $\phi \to e^{i\alpha}\phi$, which represents a $U(1)$ symmetry. Applying Noether's theorem we can write the conserved charge in differential form language:

$$j_{(1)} = i(\phi \mathrm{d}\phi^* - \phi^* \mathrm{d}\phi) \,. \tag{1.42}$$

In this language, the conserved charge is defined by integration of the current on a co-dimension 1 manifold, explicitly

$$\mathcal{Q}(\mathcal{M}_{d-1}) = \int_{\mathcal{M}_{d-1}} \star j_{(1)} \,. \tag{1.43}$$

This operator counts the number of intersections between the current as a 1-form and the co-dimension 1 surface (see Figure 1.1), in other words, it counts the times a worldline generated by the field ϕ intersects with the integration surface, which has the interpretation of counting particles, as it is expected from a $U(1)$ conserved charge. Moreover, the conservation of the current $\mathrm{d} \star j_{(1)} = 0$ implies the charge operator is topological. To see that let us consider a slightly deformed surface $\mathcal{M}'_{d-1}$ only differing from $\mathcal{M}_{d-1}$ in a limited region of spacetime (see Figure 1.2. We may then consider the difference of the charge counted in both surfaces:

$$\mathcal{Q}(\mathcal{M}'_{d-1}) - \mathcal{Q}(\mathcal{M}_{d-1}) = \int_{\mathcal{M}'_{d-1}} \star j_{(1)} - \int_{\mathcal{M}_{d-1}} \star j_{(1)} = \int_{\mathcal{S}_{d-1}} \star j_{(1)} \,, \tag{1.44}$$

being S_{d-1} the surface formed by the union of both surfaces where they do

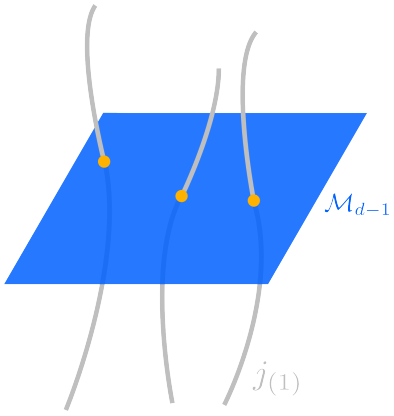


Figure 1.1: The charge operator $\mathcal{Q}(\mathcal{M}_{d-1})$ can be seen as counting the number of intersections between the manifold $\mathcal{M}_{d-1}$ where the operator is supported and the current $j_{(1)}$. In this case the $U(1)$ current can be interpreted as the worldline of a particle generated by the excitation of a field ϕ in some point, thus $\mathcal{Q}$ will count the number a worldline intersects a surface located in some point in spacetime, which is interpreted as the number of particles.

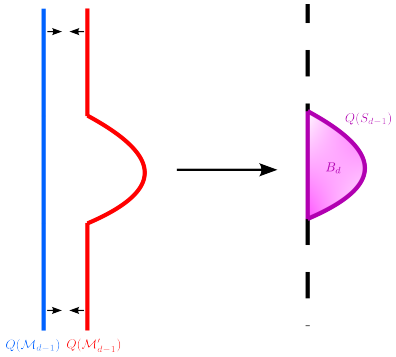


Figure 1.2: On the left: charge operator defined on the initial manifold $\mathcal{M}_{d-1}$ and on the deformed manifold $\widetilde{\mathcal{M}}_{d-1}$, separated for the pictorial representation. On the right: the difference between the charge operator define in both manifolds. The operator is canceled in the intersection, thus leaving only the operator defined on a closed manifold S_{d-1}.

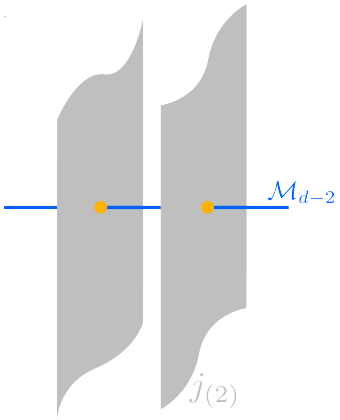


Figure 1.3: For higher-form symmetries the interpretation of the charge operator $\mathcal{Q}(\mathcal{M}_{d-1})$ is analogous. In this case, the charge operator will account for the conserved number of extended objects instead of point-like particles.

not coincide, correctly oriented. By definition of our problem, this section is a boundary $S_{d-1} = \partial B_d$, allowing us to use Stokes' theorem:

$$\mathcal{Q}(\mathcal{M}'_{d-1}) - \mathcal{Q}(\mathcal{M}_{d-1}) = \int_{B_d} \mathrm{d} \star j_{(1)} = 0\,. \tag{1.45}$$

Thus, small transformation on the integration surface do not affect on the operator, in which sense is a topological operator. The charge operator admits an exponentiation of the form

$$U_\alpha(\mathcal{M}_{d-1}) = \exp\left(i\alpha \int_{\mathcal{M}_{d-1}} \star j_{(1)}\right)\,, \tag{1.46}$$

recovering in this way the $U(1)$ transformation acting over the fields, now as a topological operator. This example not only illustrates how the symmetries on the theory can be thought as topological operators but also allows us to introduce more general structures compatible with the properties we require from a symmetry transformation. The most direct generalization is thinking on different dimensionality topological objects:

$$U_\alpha(\mathcal{M}_{d-p-1}) = \exp\left(i\alpha \int_{\mathcal{M}_{d-p-1}} \star j_{(p+1)}\right)\,. \tag{1.47}$$

This is the case of the so-called higher-form symmetries. This operators account for extended objects, such as strings (see Figure 1.3). One of the best known examples of this symmetries are the electric and magnetic symmetries of Maxwell's theory, which are 1-form symmetries [21]. All the examples we have seen so far of symmetries form the mathematical structure of a group, that is, the elements forming a symmetry are equipped with a multiplication closed under the struc-

ture of the group itself, among other properties. In the case of the example (1.46) we can realize that the composition rule of the $U(1)$ group is fullfilled

$$\begin{aligned} U_\alpha(\mathcal{M}_{d-1}) \times U_\beta(\mathcal{M}_{d-1}) &= \exp\left(i\alpha \int_{\mathcal{M}_{d-1}} \star j_{(1)}\right) \times \exp\left(i\beta \int_{\mathcal{M}_{d-1}} \star j_{(1)}\right) \\ &= \exp\left(i(\alpha+\beta) \int_{\mathcal{M}_{d-1}} \star j_{(1)}\right) = U_{\alpha+\beta}(\mathcal{M}_{d-1})\,. \end{aligned} \tag{1.48}$$

We can quickly notice that this structure defines an inverse element, in this case it is realized by selecting $\beta = -\alpha$, which is also a definitory property of a group structure. However, this is not the most general case for topological operators in general. As these are operators defined in a quantum field theory one can imagine richer structures involving other operations, in particular:

$$U_a \times U_b = \sum_c N_{ab}^c U_c\,, \tag{1.49}$$

with N_{ab}^c some coefficients. This kind of operators fulfill a mathematical structure called a fusion category ($\mathcal{C}$), defining in this way categorical or non-invertible symmetries. This structure of symmetries are typically allowed in quantum theories, where superposition allows for topological operators to fuse into a sum of them rather than just one operator. In those theories, one may find categorical structures of topological operators commuting with the Hamiltonian, making them symmetries of the quantum theory[2].

Generalized symmetries are ubiquitous in physics, thus they got a lot of interest from the community in the past years [3]. Some areas in which the study of generalized symmetries has gained interest are condensed matter [24, 25, 26], gravity [27, 28, 29], standard model [30, 31, 32] and string theory [33, 34, 35], among others.

One of our motivations to study generalized symmetries is because they provide a new way of understanding some symmetry-related phenomena. Namely, one of the main interests of this thesis, the chiral anomaly, can be understood in a completely new way by considering generalized symmetries: the chiral anomaly becomes a non-invertible symmetry in this language. In the following we will review how the chiral anomaly becomes a non-invertible topological operator, connecting it with the other main objective of the thesis, the study of categorical symmetries in quantum theories.

[2]We may notice that categorical symmetries are not represented by (anti-)unitary operators, therefore they violate Wigner's theorem. This was recently studied in [22], where it was recovered a sense of probability preservation by generalized symmetries beyond Wigner's theorem.

[3]For a recent review on the topic and some applications see [23].

1.2.1. Revisiting the chiral anomaly as a non-invertible symmetry

In this new paradigm, the chiral anomaly can be recasted in terms of a non-invertible topological operator [36, 37]. Let us consider the gauge part of the covariant anomaly, that is, the axial current in a theory without gravity (1.37):

$$\mathrm{d} \star j_A = \frac{1}{4\pi^2} \mathrm{d}A \wedge \mathrm{d}A \,. \tag{1.50}$$

This Ward identity can be rewritten to consider a conserved current:

$$\mathrm{d} \star \tilde{J} = \mathrm{d} \left(\star j_A - \frac{1}{4\pi^2} A \wedge \mathrm{d}A \right) = 0 \tag{1.51}$$

However, we may notice by definition that $\tilde{J}$ is not invariant under gauge transformations of the $U(1)$ field. However, this can be fixed at a quantum level. Let us consider the $U(1)$-valued operator associated to this conserved current:

$$U_\alpha(\mathcal{M}_3) = \exp\left(i\alpha \int_{\mathcal{M}_3} \star j_A - \frac{1}{4\pi^2} A \wedge \mathrm{d}A \right) . \tag{1.52}$$

This operator is defined on a co-dimension 1 manifold and we may notice that is not gauge invariant under large gauge transformations for a general $\mathcal{M}_3$. This happens in particular for manifolds with non-trivial 1-cycles. In order to be invariant under large gauge transformations, the parameter α has to be quantized as $\alpha \in 2\pi\mathbb{Z}$. However, it is possible to recover the $U(1)$-valued operator. To do so we may introduce a scalar field θ defined in the co-dimension 1 manifold such that, under $U(1)$ gauge transformations behaves:

$$\begin{aligned} \delta_\lambda A &= \mathrm{d}\lambda \,, \\ \delta_\lambda \theta &= \lambda \,. \end{aligned} \tag{1.53}$$

With such field, we may redefine our current to make it completely gauge invariant:

$$\tilde{U}_\alpha(\mathcal{M}_3) = \exp\left(i\alpha \int_{\mathcal{M}_3} \star j_A - \frac{1}{4\pi^2} A \wedge \mathrm{d}A + \frac{1}{4\pi^2} \mathrm{d}\theta \wedge \mathrm{d}A \right) . \tag{1.54}$$

Lastly, in order to recover the original meaning of axial transformation, we may integrate out the auxiliary field introduced on the manifold, thus leading to

$$\hat{U}_\alpha(\mathcal{M}_3) = \int [\mathcal{D}\theta] \exp\left(i\alpha \int_{\mathcal{M}_3} \star j_A - \frac{1}{4\pi^2} A \wedge \mathrm{d}A + \frac{1}{4\pi^2} \mathrm{d}\theta \wedge \mathrm{d}A \right) . \tag{1.55}$$

Due to the introduction of the path integral with the purpose of integrating out the auxiliary field, we may easily check that the $U(1)$ group structure is broken

$$U_\alpha(\mathcal{M}_3) \times U_\beta(\mathcal{M}_3) = \int [\mathcal{D}\theta \mathcal{D}\theta'] \exp\left(i(\alpha+\beta) \int_{\mathcal{M}_3} \star j_A - \frac{1}{4\pi^2} A \wedge \mathrm{d}A \right) \times \exp\left(\frac{i}{4\pi^2} \int_{\mathcal{M}_3} (\alpha \mathrm{d}\theta + \beta \mathrm{d}\theta') \wedge \mathrm{d}A \right) \neq \times U_{\alpha+\beta}(\mathcal{M}_3)\,. \tag{1.56}$$

In particular, this new structure implies that an inverse operator does not necessarily exists for every operator:

$$\nexists\, \beta \; : \quad U_\alpha(\mathcal{M}_3) \times U_\beta(\mathcal{M}_3) = \mathbb{1} \quad \forall \alpha\,. \tag{1.57}$$

Making this a non-invertible symmetry transformation.

Although we can rewrite the chiral anomaly as a non-invertible operator, we do not have many examples of non-invertible symmetries in theories with dimensionality $d > 2$. Recent works have made progress in study and describe categorical symmetries in higher dimensions, specially in $d = 3$ [38]. In 2-dimensional theories, non-invertible symmetries have been much better explored, having several examples of them. This symmetries are ubiquitous in conformal field theories. As 2-dimensional conformal field theories present a well-posed setup to study non-invertible symmetries, we will focus in those for the purpose of this thesis. In the following section we will review how categorical symmetries appear in this theories and the basic properties we need for their study.

1.2.2. Non-invertible symmetries in 2-dimensional conformal field theories

A special case of interest is that of 2-dimensional conformal field theories. In those, due to the low dimensionality, one can only build 1-dimensional operators implementing symmetries. For this reason, only 0-form symmetries, supported in lines, exist. However, such theories allow allow for categorical structure in their topological sectors. In general, a discrete internal symmetry in a 2-dimensional CFT [39] will be given by a group of lines satisfying:

$$\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}{}^c \mathcal{L}_c\,, \tag{1.58}$$

where the operators are labeled by the elements of the category $a \in \mathcal{C}$, as shown in Figure 1.4. The coefficients $N_{ab}{}^c \in \mathbb{N}$ are the so-called fusion coefficients, which

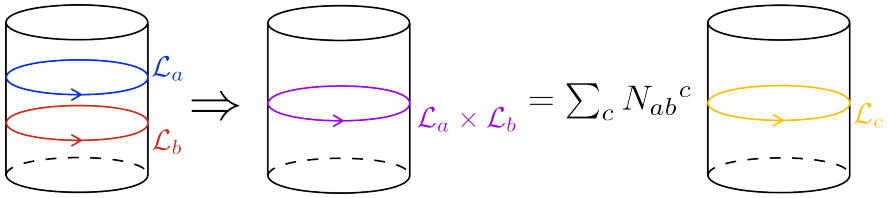


Figure 1.4: In a 1+1-dimensional CFT prepared on a cylinder, symmetry lines are acting on states of the theory when they are located along the spatial cycle. We may visualize the fusion rules between two symmetry lines as a sum of cylinder partition functions with insertions of the corresponding lines given by the product, weighted by the fusion coefficients.

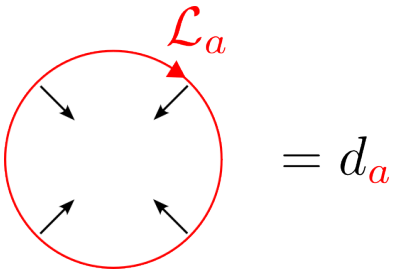


Figure 1.5: The quantum dimension can be thought as the remnant of shrinking a topological operator to a point in absence of insertions of local operators. In this sense, a unitary operator (group-like symmetry) would have quantum dimension $d_a = 1$, while non-trivial fusion rules allow for dimensions $d_a > 1$.

are typically given by the dimension of the junction vector space [39]:

$$N_{ab}{}^c = dim(V_{\mathcal{L}_a \mathcal{L}_b \bar{\mathcal{L}}_c}) \,. \tag{1.59}$$

This particular fusion structure allows for the line operator to have a quantum dimension different from one, as it is the case for group-like symmetries. We define the quantum dimension as the vacuum expectation value (see Figure 1.5):

$$\dim a \equiv d_a = \langle \mathcal{L}_a \rangle \,. \tag{1.60}$$

There exists a class of CFTs in where there is a explicit construction for these symmetry lines. In diagonal rational CFTs (diagonal RCFTs) [4], symmetry lines are given by the so-called Verlinde lines [41]. These lines are in one-to-one correspondence with the primaries of the CFT $\{\phi_a\}$. Their action is defined through the modular S matrix:

$$\mathcal{L}_a \, |\phi_b\rangle = \frac{S_{ab}}{S_{1b}} \, |\phi_b\rangle \ , \tag{1.61}$$

[4]Rational conformal field theories are a special class of CFTs where the number of primaries is finite. Some notorious examples are minimal models. These theories are called diagonal when the partition function is written as a diagonal sum of square of characters (see chapter 7 of [40]).

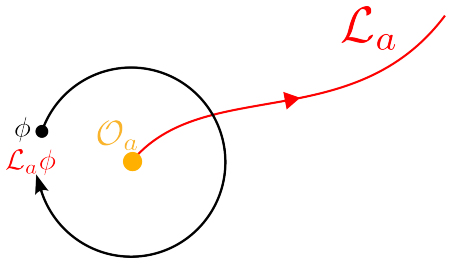


Figure 1.6: Verlinde lines define defect or twisted Hilbert spaces at their endpoints. The so-called non-genuine or defect operators belong to these spaces. These are local operators with a topological tail attached, representing the corresponding symmetry line, such that when you circle them you get a symmetry phase.

and they follow the same fusion rules as the OPEs of the primaries, which are given by the Verlinde formula:

$$N_{ab}{}^{c} = \sum_{d} \frac{S_{ad} S_{bd} S^{*}_{cd}}{S_{1d}} \,. \tag{1.62}$$

For this theories, the quantum dimensions of the topological operators (1.60) can also be written in terms of the modular S matrix:

$$d_a = \frac{S_{1a}}{S_{11}} \,. \tag{1.63}$$

In the absence of local operators, Verlinde lines are either closed lines or they extend to infinity. The only possibility for these lines to end in a specific point is to have the insertion of a local operator there. In other words, Verlinde lines define a Hilbert space at their endpoints $\mathcal{H}_{\mathcal{L}_a}$. This space is typically known as defect or twisted Hilbert space [42], and operators forming part of it are the so-called non-genuine operators. Non-genuine operators are defined in opposition to genuine operators, which are those without lines attached, and they implement the symmetry action when one goes around them as shown in Figure 1.6.

A very useful feature of symmetries is that they allow us to classify operators in a theory, that is, symmetries organize the Hilbert space of a theory. When the symmetry follows a group structure, operators are classified as transforming in different representations of the group. When the symmetry is categorical we need an analogue for representations of the group.

Finding representations of categorical symmetries might be a hard task. To carry it out it is quite useful to formulate the theory with one extra dimension. In this dimension, typically given by an interval, we isolate the symmetry structure, so that we can study it without worrying about the physical operators. This is usually known as the Symmetry Topological Field Theory (SymTFT) [20, 43, 44, 45, 46] or sandwich construction (see Figure 1.7). The formulation requires to

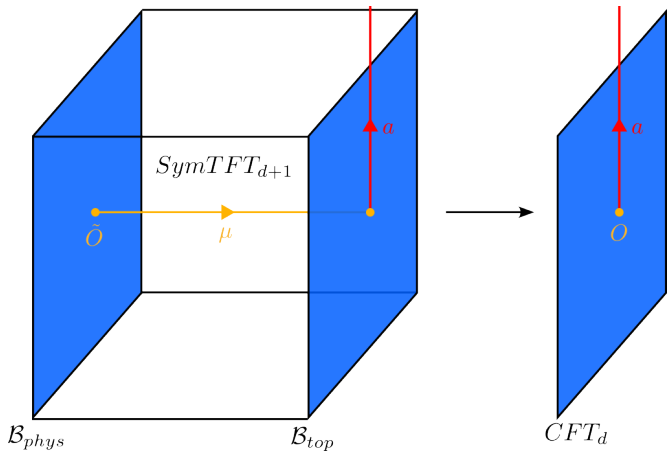


Figure 1.7: The SymTFT formulation proposes to specify a topological field theory in one extra dimension to study the global symmetries of a theory. On the left: the full SymTFT construction. It consist of a topological field theory in an interval with two boundaries: the symmetry boundary in which only topological boundary conditions are imposed and the physical boundary containing all the non-topological degrees of freedom of the original theory. On the right: our subject of study theory, which is equivalent to the SymTFT on the left after an interval compactification.

specify a topological theory in the extra dimension and put two boundaries, one purely topological and the other containing all the physical degrees of freedom. In this formulation we recover our original theory when we perform an interval compactification or "close the sandwich".

In the SymTFT the physical boundary contains all the physical degrees of freedom of the original theory, while in the symmetry boundary one can choose topological boundary conditions. As we will comment later, the choice of boundary conditions is related to symmetry breaking phases.

In 1+1-dimensional CFTs with symmetries described by a fusion category $\mathcal{C}$, the topological field theory in the bulk of the SymTFT carries a symmetry given by the Drinfeld center[5] of the category [47]. This topological field theory was first constructed in [43, 48] previously to the SymTFT formalization. In that construction it was observed that any 2-dimensional QFT with a fusion category $\mathcal{C}$ can be lifted to a boundary condition for a 3-dimensional topological field theory with a topological Dirichlet boundary condition on the other boundary. This theory is referred to as the Turaev-Viro $\text{TV}_{\mathcal{C}}$ theory, which became the SymTFT in the modern language. In that theory, symmetries are implemented by anyons μ that form a modular fusion category given by the Drinfeld center, or quantum

[5]The Drinfeld center $\mathcal{D}(\mathcal{C})$ of a category $\mathcal{C}$ is a braided fusion category that is built from $\mathcal{C}$.

double, $\mathcal{D}(\mathcal{C})$ [49, 50]. In general, the Drinfeld center satisfies:

$$\sum_{\mu\in\mathcal{D}(\mathcal{C})}(\dim\mu)^2=\left|\sum_{a\in\mathcal{C}}(\dim a)^2\right|^2. \tag{1.64}$$

In particular, for RCFTs, the quantum double takes the form:

$$\mathcal{D}(\mathcal{C})=\mathcal{C}\boxtimes\bar{\mathcal{C}}\,. \tag{1.65}$$

This particular structure means that the simple objects defined in the SymTFT are anyons of the form:

$$\mu=c\otimes\bar{d}\,, \tag{1.66}$$

with $c,d\in\mathcal{C}$ simple objects in the category. One famous example of this construction happens when the category $\mathcal{C}$ is given by a discrete group. In that case, the topological QFT is the Dijkgraaf-Witten (DW) gauge theory [51]. In particular, for the discrete group $G=\mathbb{Z}_2$ without anomaly, the DW theory is just the toric code [52]. We will explore in detail this example later.

This language is particularly useful as we can study different aspects of the symmetry by building the SymTFT with different anyon configurations. In particular, regarding representations of a category, the usual analogue to group theory are the representations of the so-called tube algebra $\mathrm{Tub}(\mathcal{C})$, which encodes all possible actions of the symmetry lines of a category $\mathcal{C}$ on every genuine and non-genuine operators in a theory[6] [47]. The irreducible representations (irreps) of $\mathrm{Tub}(\mathcal{C})$ are associated with generalized charges [53, 54]. In the SymTFT, irreps of this algebra are in one-to-one correspondence with anyons of the Drinfeld center [47], making it possible to find generalized charges and the structure of the irreps associated to them preparing the SymTFT with certain anyon configurations.

The SymTFT is not only useful to study irreps and generalized charges, it is also useful to study symmetry breaking patterns. In order to classify different breaking patterns one need to set the physical boundary also to a topological boundary condition [55, 56]. Then, as stated above, the choice of boundary conditions for the fields in the TQFT will select the symmetry breaking pattern. For the group-like symmetry case, Dirichlet boundary conditions are chosen for the topological boundary. Then, selecting between Dirichlet or Neumann conditions for the physical boundary will have, in this construction, implications regarding the symmetry breaking in the theory after closing the sandwich. If one chooses Neumann boundary conditions, the symmetry will be unbroken, while choosing Dirichlet corresponds to completely break the symmetry spontaneously. Some-

[6] In 1+1-dimensional CFTs, by state operator correspondence, this is equivalent to account for every possible configuration of lines inserted on the theory quantized on a cylinder, taking in account every possible (non-)trivial junction.

	$\mathbb{1}$	e	m	f
$\mathbb{1}$	$\mathbb{1}$	e	m	f
e		$\mathbb{1}$	f	m
m			$\mathbb{1}$	e
f				$\mathbb{1}$

Table 1.1: There are a total of four anyons in the toric code. These anyons follow the multiplication rule of an invertible symmetry, given by $\mathcal{D}(\mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

times intermediate boundary conditions is possible, in that case we say part of the symmetry will be spontaneously broken, but not the whole symmetry. As we will see in the following section, this has a very important interpretation, as the breaking patterns of a symmetry correspond to the study of different gapped phases, or in other words, different phases of matter [55, 56, 57, 58]. Now, in order to illustrate the SymTFT construction and see how the choices of boundary conditions give different phases we will review an example: the toric code.

Example: $\mathbb{Z}_2$ symmetry and the toric code

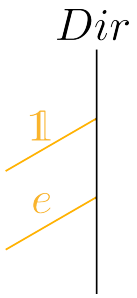
One of the best known examples of SymTFT is the toric code as the SymTFT for the $\mathbb{Z}_2$ symmetry without anomaly, which describes, for example, the phases of the Ising model. The Ising model has a $\mathbb{Z}_2$ global symmetry, given by the lines $\{\mathbb{1}, \eta\}$ with fusion rules $\eta \times \eta = \mathbb{1}$. As this is a group symmetry these lines are unitary operators, so their quantum dimensions are $d_{\mathbb{1}} = d_\eta = 1$. The quantum double of this theory is easy to build following (1.65). According to this, the SymTFT is going to be a topological field theory with simple lines, or anyons, following the fusion rules of the Drinfeld center:

$$\mathcal{D}(\mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \,. \tag{1.67}$$

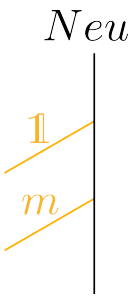
The Drinfeld center is formed by four anyons: $\{\mathbb{1}\otimes\mathbb{1}, \mathbb{1}\otimes\eta, \eta\otimes\mathbb{1}, \eta\otimes\eta\}$, which follow the multiplication rules of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group. A more usual notation for these anyons is $\{\mathbb{1}, e, m, f\}$, where the first three are bosonic and correspond with the identity, electric and magnetic anyon of the toric code and the last one is the fermionic one. The fusion rules are written in Table 1.1. As all anyons are invertible, all quantum dimensions are $d_\mu = 1$. Therefore is easy to check that (1.64) holds:

$$\sum_{\mu\in\mathbb{Z}_2\times\mathbb{Z}_2} (\dim\mu)^2 = 4 = \left|\sum_{a\in\mathbb{Z}_2} (\dim a)^2\right|^2 \,. \tag{1.68}$$

The toric code as a SymTFT for the 2-dimensional Ising model only allows for two topological boundary conditions [55, 56]. These are Dirichlet and Neumann boundary conditions. In the language of anyons, the first ones correspond to let



(a) Anyons allowed to end on a point in a boundary where Dirichlet boundary conditions have been chosen.



(b) Anyons allowed to end on a point in a boundary where Neumann boundary conditions have been chosen.

Figure 1.8: Different boundary conditions in the toric code imply that only a group of anyons can end in a point in such boundary. On the left, we can see that choosing a topological Dirichlet boundary condition only allows for the anyons $\{\mathbb{1}, e\}$ to end, while the other anyons would belong to the twisted sectors. On the right, choosing a Neumann boundary condition allows for $\{\mathbb{1}, m\}$ to end on such boundary.

the anyons $\{\mathbb{1}, e\}$ end in a point in the boundary, while the second one allows for $\{\mathbb{1}, m\}$ to end. This is shown in Figure 1.8. Now, as stated in the previous section, if one wants to look for different phases of the $\mathbb{Z}_2$ symmetry, one needs to choose for the topological boundary in the SymTFT Dirichlet boundary conditions. Then, the choice of topological boundary conditions in the physical boundary will give the different phases. Creating this concrete sandwich constructions, one is able also to obtain the number of vacua as the number of lines that are allowed to end in both boundaries of the SymTFT. In this example we have two possibilities: choosing Dirichlet or Neumann boundary conditions for the physical boundary.

By setting Dirichlet boundary conditions we find two anyons that are allowed to end in both boundaries, as shown in Figure 1.9. This correspond to the $\mathbb{Z}_2$ SSB phase. We can check that, as two anyons can end in both boundaries, there will be two vacua. In the Ising model, this phase correspond to the ordered phase, where the two vacua are identified with setting all the spins up $|\uparrow\rangle$ and all the spins down $|\downarrow\rangle$. This two vacuum states are exchanged by the action of the (spontaneously broken) group $\mathbb{Z}_2$.

The other option is choosing Neumann boundary conditions. By doing so, we find that only the identity anyon is allowed to end on both boundaries, as shown in Figure 1.10. This corresponds to the unbroken or trivial phase. This is a gapped phase with only one vacuum state, that is invariant under the symmetry. This is the case of the disordered phase of the Ising model.

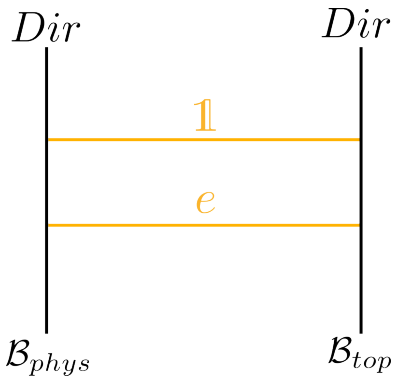


Figure 1.9: Choosing Dirichlet boundary conditions on the physical boundary allow for the identity and the electric anyon to end on both boundaries of the SymTFT. This configuration accounts for the $\mathbb{Z}_2$ SSB phase, where one can find two vacuum states.

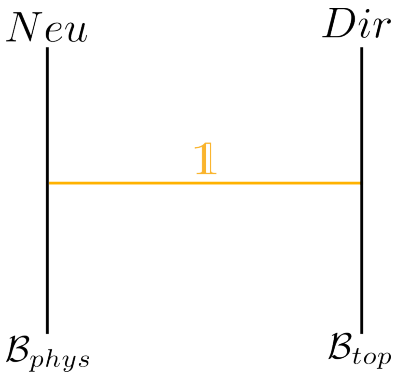


Figure 1.10: Choosing Neumann boundary conditions on the physical boundary only allows for the identity anyon to end on both boundaries. This gives a gapped phase with just one vacuum state that does not break the symmetry.

1.2.3. Symmetries and phases of matter: the Landau paradigm

Another feature of global symmetries is that they characterize phases of matter. We say a global symmetry of a theory is spontaneously broken if the ground state is not invariant under such symmetry. Spontaneous symmetry breaking (SSB) typically implies a structure of vacuum states not invariant under the action of the global symmetry but connected by it. According to the vacuum structure, we may catalog different phases of the theory. Therefore, a phase of a theory is characterize by patterns of SSB of a global symmetry. This way of characterizing phases is called the Landau paradigm [7] [60, 61, 62]. The Landau paradigm can be formulated in the following way: systems whose interactions do not preserve any symmetry only have one phase, while systems with interactions preserving a symmetry given by a group G can have more than one phases. In the latter case, different phases can be distinguished by different patterns of spontaneous breaking of the symmetry G. These phases are labeled by the subgroup $H \subseteq G$ that remains unbroken. Phases labeled by different H are always separated by a phase transition. Between two phases with compatible symmetries: $H_1 \subset H_2$ it can exist a continuous phase transition. This transition is driving by an operator that transforms trivially under H_1 but not under H_2, such operator is called an order parameter.

To illustrate how the Landau paradigm works, let us turn back to the complex boson theory (1.41) with an interaction term of the form:

$$S = \int d^d x \left(\partial_\mu \phi^* \partial^\mu \phi + m^2 (\phi^* \phi) + \frac{\lambda}{4} (\phi^* \phi)^2 \right) , \tag{1.69}$$

where the coupling constant $\lambda > 0$ so that the potential is bounded from below. In this case we can distinguish two different scenarios: positive and negative m^2. In the first one there is just one minimum of the potential, corresponding to the value of the field $\langle \phi \rangle = 0 = \langle \phi^* \rangle$. In this case we only have one vacuum state and it is invariant under the global symmetry (the $U(1)$ symmetry of the complex boson). We call this the unbroken phase.

In the case $m^2 = -\mu^2 < 0$ the potential does not have a single minima but a family of them, given by the condition $\langle \phi^* \phi \rangle = \frac{2\mu^2}{\lambda} = v^2$. Thus, there is an infinite set of vacuum states that are not invariant under the symmetry, instead they are related by it. We call this case the spontaneous symmetry broken (SSB) phase [8], and it is characterized by the condensation $\langle \phi \rangle = v$.

Not every phase of matter can be classified by the SSB of an ordinary symmetry. The old Landau paradigm works well in general for classical phases of

[7] See [59] for a modern formulation.

[8] In this case we may observe that this phase corresponds to the symmetry being completely broken spontaneously. Thus in this theory we distinguish between two phases, the unbroken one with $G = U(1)$ and the completely broken with $H = \mathbb{1}$.

matter or when the quantum phases are classical-like. This means when the state can be described without entanglement. We find phases and transitions beyond the Landau paradigm when entanglement starts playing a major role [59]. One of the best example of states where entanglement plays a crucial role are the topological phases [63, 64, 65]: Symmetry Protected Topological (SPT) phases and topological order. SPT phases are gapped phases with a global symmetry that is not spontaneously broken. They are characterized by gapless edge modes preserved by the symmetry and a short-range entanglement structure[9] [66]. On the other hand, topological order refers to gapped phases with long-range entanglement[10][67, 68, 69]. Some examples of topological phases are the fractional quantum Hall effect [70], high temperature superconductors [71, 72] and topological insulators [73, 74, 75].

For many years, it was thought that these phases lied outside the Landau paradigm. However, with the study of generalized symmetries, the old Landau paradigm was expanded to include them. This is known as the generalized Landau paradigm, and it can be formulated in the following way [59]: topological phases can be described as symmetry breaking phases of higher-form symmetries, with transitions induced by fluctuation of higher-dimensional operators. SPT phases can be mapped to symmetry breaking phases of generalized symmetries under gauging.

In this way, generalized symmetries present a new approach to describe phases of matter that lied outside the original Landau paradigm [55, 56, 57, 58]. Some examples that lay outside this paradigm are phases of gauge theories, topological order and, although not related in principle with quantum theories, phases of gravity.

As we have discussed in this section, most of the interesting effects associated to generalized symmetries, specially categorical symmetries, arise when considering systems where quantum correlations become important. In particular, non-invertible symmetries and beyond Landau phases are mainly found in these theories. Therefore, the study of the entanglement structure becomes a very interesting subject to characterize non-invertible symmetries and phases outside the classical Landau paradigm, like topological phases, in quantum theories. In the next section we will introduce the basic concepts of entanglement and how to incorporate symmetries, which will lead to the introduction of the main concepts presented in the rest of papers conforming this compendium.

[9]This is, the state can be deformed smoothly to a product state. However, in order to do so, one needs to break the symmetry protecting the edge modes.

[10]Contrarily to short-range, long-range entanglement refers to an state that cannot be deformed into a product state. The only possible way to reach a product state from a long-range entangled one is going through a gap closing phase transition.

1.3 | Entanglement entropy

As stated in the last section, non-invertible symmetries appear naturally in quantum theories, where entanglement plays a fundamental role. Entanglement is a fundamental concept of quantum theories as it characterizes non-local correlations only existing in these theories due to superposition. The entanglement structure of a state in a quantum theory is characterized by a quantity called entanglement entropy. Let us start by reviewing the basics of entanglement in our theories of interest: 1+1-dimensional conformal field theories [11].

In a quantum system, the information carried by a state is described by its density matrix. For a pure state $|\psi\rangle$, the density matrix $\rho = |\psi\rangle\langle\psi|$ is the quantum equivalent to the classical probability distribution. The amount of information carried by a state is define as the von Neumann entropy of its density matrix:

$$S_{vN}(\rho) = -\mathrm{Tr}\rho\log\rho\,. \tag{1.70}$$

In a 1+1-dimensional conformal field theory, the density matrix corresponding to the ground state can be defined in terms of an Euclidean path integral:

$$\rho = |\psi\rangle\langle\psi| = \int_{\phi(\tau=0^-)=\bullet}^{\phi(\tau=0^+)=\bullet} [\mathcal{D}\phi]_{\mathbb{C}}\, e^{-S[\phi]}\,, \tag{1.71}$$

where $\bullet$ represent unspecified boundary conditions, in this context, we leave open the choice of field configuration at the time slices corresponding to $\tau = 0^+$ and $\tau = 0^-$. Initial and final boundary conditions, those are, $\phi(\tau = -\infty)$ and $\phi(\tau = \infty)$ need to be specified in principle to define the path integral. However, these boundary conditions are going to represent just a normalization constant, so we will drop them through all the text and normalize correctly the states. This definition of density matrix (1.71) admits a pictorial representation, shown in Figure 1.11. In this language, the matrix elements of ρ are defined by specifying field configurations as boundary conditions:

$$\langle\phi_1|\,\rho\,|\phi_2\rangle = (\rho)_{12} = \int_{\phi(\tau=0^-)=\phi_1}^{\phi(\tau=0^+)=\phi_2} [\mathcal{D}\phi]_{\mathbb{C}}\, e^{-S[\phi]}\,. \tag{1.72}$$

The product of two density matrices can be defined as a regular product of matrix elements. For density matrices defined in this way, the product requires to insert field configurations and path integrating over them:

$$(\rho\times\sigma)_{ij} = \sum_k (\rho)_{ik}(\sigma)_{kj} \equiv \int [\mathcal{D}\phi_k]\,\langle\phi_i|\,\rho\,|\phi_k\rangle\,\langle\phi_k|\,\rho\,|\phi_j\rangle\ . \tag{1.73}$$

[11] For a modern review on entanglement we refer the reader to [76].

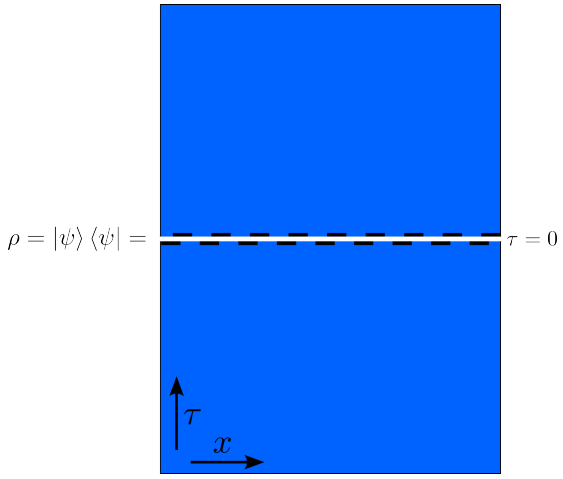


Figure 1.11: The density matrix of a pure state can be defined using Euclidean path integral. This method admits pictorial representation. Euclidean path integrals in 2 dimensions can be visualized as the complex plane $\mathbb{C}$ with, maybe, some insertion of operators. When there are no insertion of operators, we say the path integral is prepared on the ground state of the theory. In our pictorial notation, dashed lines correspond to unspecified field configuration in the corresponding region.

This is known as sewing the path integral. By doing this process we are allowed to define interesting quantities such as the square of a density matrix:

$$\rho^2 = \int [\mathcal{D}\phi_k] \int_{\phi(\tau_1=0^-)=\bullet}^{\phi(\tau_1=0^+)=\phi_k} [\mathcal{D}\phi]_{\mathbb{C}}\, e^{-S[\phi]} \int_{\phi(\tau_2=0^-)=\phi_k}^{\phi(\tau_2=0^+)=\bullet} [\mathcal{D}\phi]_{\mathbb{C}}\, e^{-S[\phi]} \equiv \int_{\phi(\tau_1=0^-)=\bullet}^{\phi(\tau_2=0^+)=\bullet} [\mathcal{D}\phi]_{\mathbb{C}^2}\, e^{-S[\phi]}\,, \tag{1.74}$$

and the trace by just sewing the two unspecified boundary conditions of the path integral defining the density matrix (1.71):

$$Z = \mathrm{Tr}\rho = \int [\mathcal{D}\phi_k] \int_{\phi(\tau=0^-)=\phi_k}^{\phi(\tau=0^+)=\phi_k} [\mathcal{D}\phi]_{\mathbb{C}}\, e^{-S[\phi]} \equiv \int [\mathcal{D}\phi]_{\mathbb{C}}\, e^{-S[\phi]}\,. \tag{1.75}$$

Entanglement entropy [77, 78, 79, 80] is a measure of quantum correlations between a region A and its complementary B. This quantity is defined through the reduced density matrix ρ_A, which accounts for the degrees of freedom supported in the region A. Thus, to define it, we need to split the Hilbert space into the two regions $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, so that we can trace out the degrees of freedom of the complementary part. However, in continuum field theories the Hilbert space is typically non-decomposable. This implies that the Hilbert space is, in general, not decomposable, thus we are unable to define a trace operation. To be able to define entanglement quantities we have to give a prescription for the decomposition of the Hilbert space. One possible prescription [81, 82] is defining the map

$$\iota_\epsilon : \mathcal{H} \to \mathcal{H}_A^{\alpha\beta} \otimes \mathcal{H}_B^{\beta\alpha}\,, \tag{1.76}$$

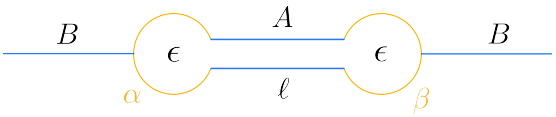


Figure 1.12: One possible prescription to split the Hilbert space in two regions is by creating two disks of radius ϵ at the separation points between the two regions, then imposing boundary conditions on them.

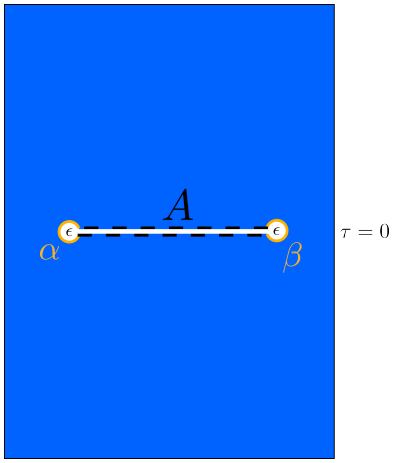


Figure 1.13: From the path integral definition, one may visualized the reduced density matrix ρ_A as the vacuum density matrix where we have traced out over the complement part of the interval B and left unspecified boundary conditions in A.

that takes the Hilbert space to a decomposable one with some boundary conditions imposed at the entangling surface. This can be visualized as defining two disks of radius ϵ at the entangling dots, where we impose simple boundary conditions (see Figure 1.12). Then ϵ acts as a UV regulator for the entanglement entropy. With this map we may define the reduced density matrix as:

$$\rho_A^{\alpha\beta} = \mathrm{Tr}_{\mathcal{H}_B^{\beta\alpha}}(\iota_\epsilon \rho \iota_\epsilon^\dagger)\,. \tag{1.77}$$

From now on we will drop the boundary conditions labels for the sake of notation simplicity. We may write the reduced density matrix as a path integral of the form:

$$\rho_A = \int_{\phi(\tau=0^-,\,x=A)=\bullet}^{\phi(\tau=0^+,\,x=A)=\bullet} [\mathcal{D}\phi]_{\mathbb{C}}\, e^{-S[\phi]}\,, \tag{1.78}$$

where we have implicitly assumed the prescription for partitioning the Hilbert space in the path integral, as shown in Figure 1.13.

Entanglement entropy is defined as the von-Neumann entropy of the reduced density matrix:

$$S_{EE}(\rho) = -\mathrm{Tr}\rho_A \log \rho_A\,. \tag{1.79}$$

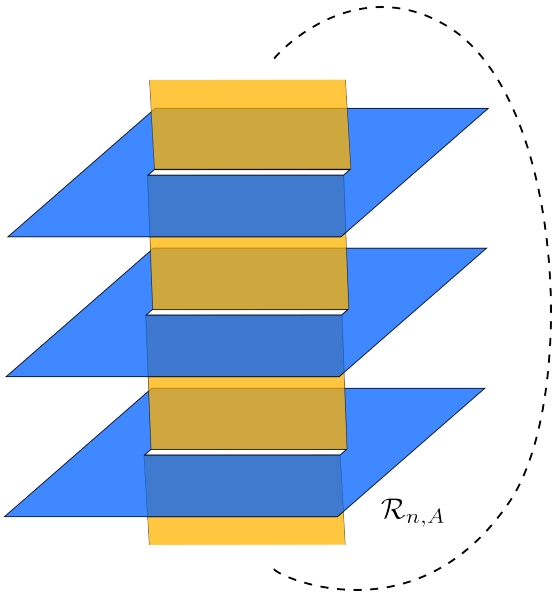


Figure 1.14: After sewing path integrals to define the replicated partition function $Z_A^{(n)}$, one lands on the replica manifold $\mathcal{R}_{n,A}$. This multi-sheeted manifold has some singular features: there are branch cuts at the places corresponding to the region A and conical singularities at the entangling dots. The replica manifold also counts with a global symmetry, the replica symmetry, corresponding to the permutation of sheets on the manifold.

Computing entanglement entropy usually requires the replica trick. In that setup, entanglement entropy is obtained as the limit $n \to 1$ of the Rényi entropies:

$$S_{EE}(\rho) = \lim_{n\to 1} S^{(n)}(\rho) = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr}\rho_A^n \,. \tag{1.80}$$

Rewriting entanglement entropy in terms of the Rényi entropies is specially useful, as the full computation reduces to evaluate the replicated partition function $Z_A^{(n)} = \mathrm{Tr}\rho_A^n$. This partition function involves taking the n-th power of the reduced density matrix. This can be constructed from (1.78) by sewing consecutively path integrals in the same fashion of (1.74). However, as we are now sewing intervals of a time slice, we are going to obtain a more complicated manifold after the process of sewing. The result of this operation is known as the replica manifold, shown in Figure 1.14. Under some conformal transformations, this geometry can be mapped into an annulus [81], with circumference $2\pi n$ and width

$$W = 2\log(\ell/\epsilon) + \mathcal{O}(\epsilon)\,, \tag{1.81}$$

and where the conformal boundary conditions appear as the boundaries of the annulus geometry (see Figure 1.15. In the annulus geometry the replica partition

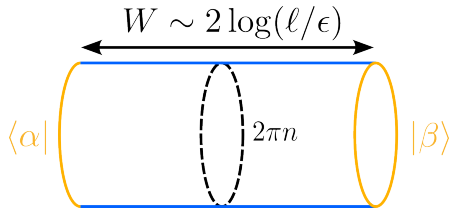


Figure 1.15: The replica partition function may be written as an amplitude using the annulus geometry. In this geometry, the boundary conditions appear explicitly as states defined in the boundaries of the annulus.

function take a simple form, it can be read as an amplitude of the form:

$$Z_A^{(n)} = \mathrm{Tr}[q^{n(L_0-c/24)}] = \langle\alpha|\, \tilde{q}^{\frac{1}{n}(L_0-c/24)}\, |\beta\rangle \ , \tag{1.82}$$

where the two last steps are related through a modular S transformation. This way of writing the partition function allows us to compute it as an amplitude, where $q = e^{-2\pi^2/W}$ ($\tilde{q} = e^{-2W} \sim e^{-4\log \ell/\epsilon}$) is the nome, L_0 is the 0-th Virasioro generator and c the central charge of the CFT. We may notice that the boundary conditions α and β appear in an explicit way in the partition function as states in the boundaries of the annulus geometry (see Figure 1.15). The states chosen have to respect the conformal symmetry, those are simple boundaries in the BCFT. In diagonal RCFTs, simple boundaries can be fully constructed using the modular data of the theory via the Cardy construction [83]:

$$|c_i\rangle = \sum_j \frac{S_{ij}}{\sqrt{S_{1j}}} |j\rangle\rangle \, , \tag{1.83}$$

where $|i\rangle\rangle$ denotes the Ishibashi states, which form a basis of the boundary CFT and are found as a solution to :

$$(L_n - \bar{L}_{-n})\, |i\rangle\rangle = 0 \, . \tag{1.84}$$

At leading order in ϵ/ℓ, which corresponds to the $q \to 1$ ($\tilde{q} \to 0$), the annulus partition function (1.82) can be evaluated in a very simple way. By inserting a complete set of eigenstates $L_0\,|i\rangle = h_i\,|i\rangle$:

$$\langle\alpha|\, \tilde{q}^{\frac{1}{n}(L_0-c/24)} \left(\sum_i |i\rangle\, \langle i| \right) |\beta\rangle = \langle\alpha|i\rangle\langle i|\beta\rangle \tilde{q}^{\frac{1}{n}(h_i-c/24)} \sim g_\alpha g_\beta \tilde{q}^{\frac{-c}{24n}} \, , \tag{1.85}$$

where the leading order is given by the smallest eigenvalue $L_0\,|0\rangle = 0$, that is, the vacuum state. We have defined $g_\alpha = \langle\alpha|0\rangle$. To leading order, the Rényi entropies

read:

$$S_A^{(n)} \sim \frac{-c}{24}\left(\frac{1+n}{n}\right)\log\tilde{q} + \log g_\alpha + \log g_\beta \sim \frac{c}{6}\left(\frac{1+n}{n}\right)\log\frac{\ell}{\epsilon} + \log g_\alpha + \log g_\beta\,, \tag{1.86}$$

and by taking the entanglement entropy limit ($n \to 1$)

$$S_{EE} \sim \frac{c}{3}\log\frac{\ell}{\epsilon} + \log g_\alpha + \log g_\beta\,. \tag{1.87}$$

In this expression we can identify at leading order the universal term for entanglement entropy in 2-dimensional CFTs. We also obtained a scheme-dependent term at $\mathcal{O}(\epsilon^0)$. We identify this term as the Affleck-Ludwig boundary entropy [84].

1.3.1. Symmetry resolved entanglement entropy

In the presence of a global symmetry, it is possible to define conserved charges (for example, the Noether charge (1.43) for continuous symmetries). Conserved charges may help us organize the Hilbert space of a theory. In general, it is possible to decompose the Hilbert space into charge sectors:

$$\mathcal{H} = \bigoplus_{\mathcal{Q}} \mathcal{H}_{\mathcal{Q}}\,. \tag{1.88}$$

For each charge sector, there exists a projection operator

$$P_{\mathcal{Q}} : \mathcal{H} \longrightarrow \mathcal{H}_{\mathcal{Q}}\,, \tag{1.89}$$

such that the density matrix of the ground state in the original Hilbert space can be decomposed into charge sectors:

$$\rho = \bigoplus_{\mathcal{Q}} [P_{\mathcal{Q}}\rho P_{\mathcal{Q}}^\dagger] = \bigoplus_{\mathcal{Q}} [P_{\mathcal{Q}}\rho] = \sum_{\mathcal{Q}} p[\mathcal{Q}]\rho[\mathcal{Q}]\,, \tag{1.90}$$

where we have used that the ground state is symmetric under the action of the global symmetry $[P_{\mathcal{Q}}, \rho] = 0$ and defined $p[\mathcal{Q}]$ as the probability of being in a determined charge sector:

$$p[\mathcal{Q}] = \mathrm{Tr} P_{\mathcal{Q}}\rho\,. \tag{1.91}$$

In this construction, we may wonder how entanglement entropy is distributed among the charged sectors associated to a global symmetry. This question is only well posed if the decomposition into charged sectors (1.88) is compatible with our prescription for splitting the Hilbert space into two regions (1.76). This

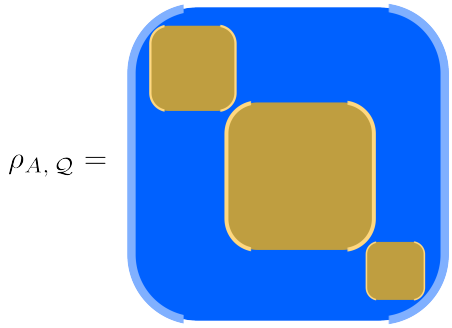


Figure 1.16: We can write the reduced density matrix in a block-diagonal form related to the symmetry structure. Each of the blocks in which the matrix is divided is a reduced density matrix projected into a charge sector of the global symmetry. Symmetry resolved entanglement entropy measures how much entanglement entropy is carried by each of these blocks.

is, the charges have to be compatible with the Hilbert space splitting:

$$\mathcal{Q} = \mathcal{Q}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \mathcal{Q}_B \,. \tag{1.92}$$

This decomposition not being possible would be a signal of the symmetry not holding at a local level in our theory. This is a signature of a 't Hooft anomaly. If this decomposition holds, we can define the projected reduced density matrix as:

$$\rho_A = \bigoplus_{\mathcal{Q}} [P_{\mathcal{Q}} \rho_A] = \sum_{\mathcal{Q}} p[\mathcal{Q}] \rho_A[\mathcal{Q}] \,. \tag{1.93}$$

We can see this matrix as a block-diagonal density matrix (see Figure 1.16). Symmetry resolved entanglement entropy [85] is then defined as the von-Neumann entropies of each projected reduced density matrices:

$$S_A[\mathcal{Q}] = -\mathrm{Tr}\rho_A[\mathcal{Q}] \log \rho_A[\mathcal{Q}] \,, \tag{1.94}$$

which can be computed using the replica trick with the projected partition function:

$$\tilde{Z}_A^{(n)}[\mathcal{Q}] = \mathrm{Tr}[\rho_A^n[\mathcal{Q}]] = \frac{\mathrm{Tr}[P_{\mathcal{Q}} \rho_A^n]}{(\mathrm{Tr}[P_{\mathcal{Q}} \rho_A])^n} \equiv \frac{Z_A^{(n)}[\mathcal{Q}]}{(Z_A[\mathcal{Q}])^n} \,, \tag{1.95}$$

where we have defined $Z_A[\mathcal{Q}] = \mathrm{Tr}(P_{\mathcal{Q}} \rho_A)$ as the unnormalized projected partition function. Symmetry resolved Rényi entropies are defined:

$$S_A^{(n)}[\mathcal{Q}] = \frac{1}{1-n} \log \frac{Z_A^{(n)}[\mathcal{Q}]}{(Z_A[\mathcal{Q}])^n} \,. \tag{1.96}$$

For ordinary group-like symmetries, charge sectors are given by the irreducible representations (irreps) of the group. For discrete groups, the projectors (1.89)

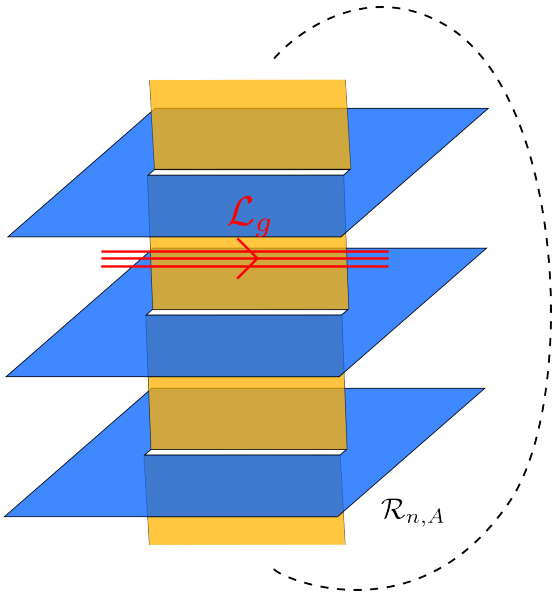


Figure 1.17: A charged moment corresponds to the replicated partition function when a topological operator is inserted in it. In the replica manifold, these are usually interpreted as inserting a flux (or symmetry element) between two consecutive sheets.

into irreducible representations take the simple form:

$$P_{\mathcal{Q}} \equiv P_r = \frac{d_r}{|G|} \sum_{g \in G} \chi_r^*(g) \xrightarrow{\widehat{\mathcal{L}}_g} , \tag{1.97}$$

where d_r is the dimension of the irrep, $|G| = \sum_r d_r^2$ is the order of the group, $\chi_r^*(g)$ is the character of the group element g in the irrep r and the line $\mathcal{L}_g$ represents the topological operator implementing the action of the group element g. Straightforwardly, the replicated projected partition function (1.95) can be written as:

$$Z_A^{(n)}[\mathcal{Q}] \equiv Z_A^{(n)}[r] = \frac{d_r}{|G|} \sum_{g \in G} \chi_r^*(g) Z_A^{(n)}[g] \,, \tag{1.98}$$

where we have defined the charged moments:

$$Z_A^{(n)}[g] = \mathrm{Tr}[\mathcal{L}_g \rho_A^n] \,. \tag{1.99}$$

The charged moments correspond to the replicated partition function with the insertion of one single topological operator representing a symmetry, as shown in Figure 1.17. These are typically the building blocks of the symmetry resolved Rényi entropies, thus computing entanglement entropy is reduced to the calculation of the charged moments for every topological operator representing the symmetry. It might be more convenient, as shown in the previous section, to use

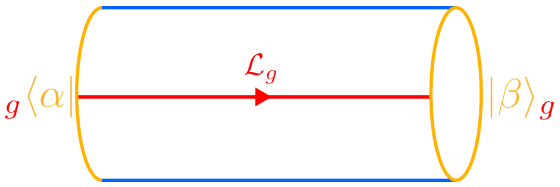


Figure 1.18: In the annulus partition function, the fluxes inserted to define charged moments are represented by symmetry lines ending on the boundary states. These lines can be interpreted as twisting the boundary state, thus the boundary conditions chosen for each charged moment must come from the corresponding twisted sector.

the annulus partition function for the computation of the charged moments.

$$Z_A^{(n)}[g] = \mathrm{Tr}[\mathcal{L}_g\, q^{n(L_0-c/24)}] = {}_g\langle\alpha|\, \tilde{q}^{\frac{1}{n}(L_0-c/24)}\, |\beta\rangle_g\,, \tag{1.100}$$

where the charged moments are now interpreted as amplitudes with twisted boundary conditions $|\alpha\rangle_g$ and $|\beta\rangle_g$, as shown in Figure 1.18. For this construction to hold, the boundary conditions have to respect the global symmetry we are resolving with respect to. This is, the symmetry line has to end topologically on the boundaries of the region A. For group-like symmetries, topological endability is equivalent to invariance. It might be the case that there does not exist an invariant state in our theory so that we cannot impose topological endability. We can see that case as a failure in finding a decomposition of the form 1.92, and so as the global symmetry of the theory cannot be introduce in the interval. This is a signature of a 't Hooft anomaly. Therefore, anomalies manifest in the boundary CFT construction as the non-existence of symmetry-invariant simple boundary conditions [86, 87]. In this approach, the anomaly forbids symmetry resolution.

If symmetric boundary conditions exist, each charged moment contributes to leading order as:

$${}_g\langle\alpha|\, \tilde{q}^{\frac{1}{n}(L_0-c/24)}\, |\beta\rangle_g \sim g_\alpha^g g_\beta^g \tilde{q}^{-\frac{c-24h_0^g}{24n}}\,, \tag{1.101}$$

with h_g^0 the conformal weight of the lowest conformal dimension primary $|\Delta_g\rangle$ in the twisted space $\mathcal{H}_g$, that is, the conformal weight of the twisted vacuum $L_0\,|\Delta_g\rangle = h_0^g\,|\Delta_g\rangle$. The g-factors are numbers defined as $g_\alpha^g = {}_g\langle\alpha|\Delta_g\rangle$. From (1.101), we may notice that, compared to the identity, any charged moment behaves at leading order as:

$$\frac{Z_A^{(n)}[g]}{Z_A^{(n)}[e]} \sim \frac{g_\alpha^g g_\beta^g}{g_\alpha g_\beta} \tilde{q}^{\frac{24h_0^g}{24n}} \xrightarrow[\tilde{q}\to 0]{} 0\,, \tag{1.102}$$

where we have taking in account that the g-factors are just numbers and $h_0^g \geq 0$, and it is only 0 for the untwisted vacuum. Thus, the identity sector dominates over all charged sectors, which gives us a general expression for the projected

partition function

$$Z_A^{(n)}[r] = \frac{d_r}{|G|}\frac{Z_A^{(n)}[e]}{(Z_A[e])^n}\chi_r^*(e)\left(1+\sum_{g\neq e}\frac{\chi_r^*(g)}{\chi_r^*(e)}\frac{Z_A^{(n)}[g]}{Z_A^{(n)}[e]}\right) \sim \frac{d_r^2}{|G|}\frac{Z_A^{(n)}[e]}{(Z_A[e])^n}\,, \quad (1.103)$$

with $d_r = \chi_r^*(e)$. Therefore, the symmetry resolved Rényi entropies will follow a universal formula for discrete groups in diagonal RCFTs:

$$S_A^{(n)}[r] \sim \frac{c}{6}\left(\frac{1+n}{n}\right)\log\frac{\ell}{\epsilon} + \log\frac{d_r^2}{|G|} + \log g_\alpha + \log g_\beta\,, \quad (1.104)$$

And finally, by taking the limit $n \to 1$, the symmetry resolved entanglement entropies:

$$S_{EE}[r] \sim \frac{c}{3}\log\frac{\ell}{\epsilon} + \log\frac{d_r^2}{|G|} + \log g_\alpha + \log g_\beta\,. \quad (1.105)$$

We may see that, for finite groups, the result is quite similar to regular entanglement entropy (1.87). There is just one $\mathcal{O}(1)$ term that distinguish between different irreps. This term contains information regarding the size of the charge sector, being equal for every irrep just in the case of abelian discrete groups. In that case, the symmetry resolved entanglement entropies do not depend on the irrep. This is known as entanglement equipartition [88].

Symmetry resolved entanglement entropy gives us information about how the information of a state is distributed along sectors related to a global symmetry. As it is a quantity connected to experimental results [89], its computation gains importance in characterizing symmetries in a system, specially extending the well-known computations for group-like symmetries [85, 87] to include generalized symmetries. Some progress on this have been recently made [90, 91, 92, 93], being some of this part of this thesis.

One of the main objectives of this thesis is studying the generalization of symmetry resolved entanglement entropy to include categorical symmetries. This will be explored in Chapters 3 and 4, where we will study for the first time the symmetry resolution for non-invertible symmetries in 1+1-dimensional conformal field theories. In particular, in Chapter 3 we will present the first computation of categorical-symmetry resolved entanglement entropy (CaT-SREE). In this chapter we will limit our studies to the untwisted sector of the theories considered. This sector, despite containing relevant information regarding symmetry resolution, does not split perfectly into charge sectors when considering categorical symmetries. This feature differentiates non-invertible symmetries from ordinary ones. In Chapter 4 we will explore this by including all twisted sectors in the symmetry resolution. In particular, we will consider resolution with respect to generalized charge sectors (or anyonic sectors) of non-invertible symmetries.

Due to the importance of SREE in this text, we will review a well-known

example next, as this will help us introduce the fundamental techniques we need to expand the study of symmetry resolved entanglement entropy to non-invertible symmetries.

Example: The critical Ising model

The critical Ising model is a 1+1-dimensional diagonal RCFT with central charge $c = \frac{1}{2}$. There are three primaries in the model $\{\mathbb{1}_{0,0}, \varepsilon_{\frac{1}{2},\frac{1}{2}}, \sigma_{\frac{1}{16},\frac{1}{16}}\}$. The modular S-matrix of the model is:

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} . \tag{1.106}$$

As this is a diagonal RCFT, the symmetry lines are in one-to-one correspondence with the primary operators. Therefore there are thre symmetry lines $\{\mathbb{1}, \eta, \mathcal{N}\}$, and their action on the primaries is given by (1.61):

$$\begin{aligned} &\mathbb{1}\,|\mathbb{1}\rangle = |\mathbb{1}\rangle\ , \quad \mathbb{1}\,|\varepsilon\rangle = |\varepsilon\rangle\ , \quad \mathbb{1}\,|\sigma\rangle = |\sigma\rangle\ , \\ &\eta\,|\mathbb{1}\rangle = |\mathbb{1}\rangle\ , \quad \eta\,|\varepsilon\rangle = |\varepsilon\rangle\ , \quad \eta\,|\sigma\rangle = -\,|\sigma\rangle\ , \\ &\mathcal{N}\,|\mathbb{1}\rangle = \sqrt{2}\,|\mathbb{1}\rangle\ , \quad \mathcal{N}\,|\varepsilon\rangle = -\sqrt{2}\,|\varepsilon\rangle\ , \quad \mathcal{N}\,|\sigma\rangle = 0\,. \end{aligned} \tag{1.107}$$

The fusion rules of these lines are in correspondence with the OPEs of the primaries, following the Verlinde formula (1.62):

$$\eta \times \eta = \mathbb{1}\,, \quad \mathcal{N} \times \eta = \mathcal{N} = \eta \times \mathcal{N}\,, \quad \mathcal{N} \times \mathcal{N} = \mathbb{1} + \eta\,. \tag{1.108}$$

The full symmetry forms a categorical structure known as a Tambara-Yamagami category $TY(\mathbb{Z}_2)$. In this category we can distinguish a subgroup $\mathbb{Z}_2 = \{\mathbb{1}, \eta\}$. As a group, there are two irreducible representations, whose projectors are:

$$P_{even} = \frac{1}{2}\ \xrightarrow{\ \mathbb{1}\ }\ + \frac{1}{2}\ \xrightarrow{\ \eta\ }\ , \tag{1.109}$$

$$P_{odd} = \frac{1}{2}\ \xrightarrow{\ \mathbb{1}\ }\ - \frac{1}{2}\ \xrightarrow{\ \eta\ }\ . \tag{1.110}$$

As the group is abelian, we may notice $d_{even} = 1 = d_{odd}$. By inserting the projectors in the partition function:

$$\begin{aligned} Z_A^{(n)}[even] =& \frac{1}{2}\frac{Z_A^{(n)}[\mathbb{1}]}{(Z_A[\mathbb{1}])^n} + \frac{1}{2}\frac{Z_A^{(n)}[\eta]}{(Z_A[\mathbb{1}])^n} \\ Z_A^{(n)}[odd] =& \frac{1}{2}\frac{Z_A^{(n)}[\mathbb{1}]}{(Z_A[\mathbb{1}])^n} - \frac{1}{2}\frac{Z_A^{(n)}[\eta]}{(Z_A[\mathbb{1}])^n} \end{aligned} \tag{1.111}$$

In order to compute the charged moments as the annulus partition function we have to find simple symmetric boundary states. In the critical Ising model one can find three simple boundaries, using the Cardy construction (1.83):

$$\begin{aligned} |\uparrow\rangle &= \frac{1}{\sqrt{2}}|\mathbb{1}\rangle\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle\rangle + \frac{1}{2^{1/4}}|\sigma\rangle\rangle\,, \\ |\downarrow\rangle &= \frac{1}{\sqrt{2}}|\mathbb{1}\rangle\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle\rangle - \frac{1}{2^{1/4}}|\sigma\rangle\rangle\,, \\ |f\rangle &= |\mathbb{1}\rangle\rangle - |\varepsilon\rangle\rangle\,, \end{aligned} \tag{1.112}$$

where the Ishibashi are named in analogy with the primaries and symmetry lines act in the corresponding way. These states have a lattice interpretation. They correspond respectively to spin up, spin down and free spin states in the Ising lattice model. The action of the $\mathbb{Z}_2$ symmetry on the states is

$$\eta\,|\uparrow\rangle = |\downarrow\rangle\ , \quad \eta\,|\downarrow\rangle = |\downarrow\rangle\ , \quad \eta\,|f\rangle = |f\rangle\ . \tag{1.113}$$

We notice that there exist one symmetric simple boundary, given by the free spin condition $|f\rangle$. Thus, in order to symmetry resolve we may prepare the annulus partition function with such boundary for the identity charged moment. Regarding the η-charged moment, we may take a look at the defect Hilbert space to find the symmetric boundaries:

$$\mathcal{H}_\eta : \{\psi_{\frac{1}{2},0}, \bar{\psi}_{0,\frac{1}{2}}, \mu_{\frac{1}{16},\frac{1}{16}}\}\,. \tag{1.114}$$

We may notice there is one scalar twisted primary $\mu_{\frac{1}{16},\frac{1}{16}}$, therefore the state associated to that primary is assured to fulfill directly the condition (1.84). As it is the only scalar, the twisted Cardy state must be proportional to it $|\mu\rangle_\eta = \alpha\,|\mu\rangle\rangle_\eta$. Moreover, as it is part of the twisted Hilbert space, it is guaranteed that it is invariant under the symmetry, therefore we can prepare the η-charged moment with the boundary condition $|\mu\rangle_\eta$. Thus, the charged moments take the form:

$$\begin{aligned} Z_A^{(n)}[\mathbb{1}] &= \langle f|\,\tilde{q}^{\frac{1}{n}(L_0 - c/24)}\,|f\rangle = \tilde{q}^{\frac{-c}{24n}} - \tilde{q}^{\frac{12-c}{24n}}\,, \\ Z_A^{(n)}[\eta] &= {}_\eta\langle\mu|\,\tilde{q}^{\frac{1}{n}(L_0 - c/24)}\,|\mu\rangle_\eta = \alpha^2\tilde{q}^{\frac{3-2c}{48n}}\,. \end{aligned} \tag{1.115}$$

We may immediately notice that, at leading order in ϵ/ℓ ($\tilde{q} \to 0$), the identity charged moment dominates over the η-charged moment. Moreover, at leading order $Z_A^{(n)}[\mathbb{1}] \sim \tilde{q}^{\frac{-c}{24n}}$. Plugging this into the partition functions (1.111), we find:

$$Z_A^{(n)}[even] \sim Z_A^{(n)}[odd] \sim \frac{1}{2}\tilde{q}^{\frac{-c}{24}\left(\frac{1}{n} - n\right)}\,. \tag{1.116}$$

Therefore, at leading order we find:

$$S_{EE}[even] = S_{EE}[odd] = \frac{c}{3}\log\frac{\epsilon}{\ell} + \log\frac{1}{2}\,, \tag{1.117}$$

so there is entanglement equipartition for the $\mathbb{Z}_2$ symmetry in the critical Ising model.

We may notice that even this model presents categorical symmetries, namely the Kramers-Wannier duality defect $\mathcal{N}$ obeys non-group-like fusion rules. As stated earlier, we will explore how to extend the calculation carried out in this example as part of the main text of the thesis, in particular in Chapters 3 and 4. In addition, we will find that anomalies play a fundamental role in this calculation, as they forbid symmetries to be compatible with splitting the Hilbert space. This will ultimately not allow us to resolve with respect to the full symmetry of the critical Ising model, for which we will review other examples where non-anomalous categorical symmetries can be found.

1.3.2. Relative entropy

Entropic quantities can also be used to study symmetry breaking patterns by probing the vacua structure, thus connecting with the (generalized) Landau paradigm. As reviewed in last section, the Landau paradigm establishes that different phases of matter can be identified by order parameters, those are operators exhibiting non-zero vacuum expectation values when a symmetry is spontaneously broken. However, order parameters may be difficult to identify in some theories. For this reason, new quantities based on information theory gained importance in the study of symmetry breaking patterns [94, 95, 96, 97]. Typically, quantities used for detecting symmetry breaking are those allowing us to compare between two density matrices representing the vacua structure. Of particular interest is the relative entropy between two states:

$$S(\rho||\sigma) = \mathrm{Tr}\rho\log\rho - \mathrm{Tr}\rho\log\sigma\,. \tag{1.118}$$

This is a positive definite quantity $S(\rho||\sigma) \geq 0$, vanishing if and only if both density matrices are the same $\rho = \sigma$. The relative entropy can also be computed using the replica trick:

$$S^{(n)}(\rho||\sigma) = \frac{1}{n-1}\log\left(\frac{\mathrm{Tr}[\rho^n]}{\mathrm{Tr}[\rho\sigma^{n-1}]}\right)\,. \tag{1.119}$$

This quantity is specially interesting because it allows us to compare how similar two density matrices are. It admits a probabilistic interpretation: having prepared our system in a state ρ and after N well-designed experimental measures, the probability of mistaking the state ρ for σ decays as $p \sim e^{-NS(\rho||\sigma)}$. One pos-

sible application is to give an entropic measure of how symmetric a state is. For that we may define a symmetrization map [94, 98]:

$$\mathcal{S} : \rho \longrightarrow \rho_{\mathcal{S}} \,, \tag{1.120}$$

with the following properties:

- Is linear: $(\alpha\rho + \beta\sigma)_{\mathcal{S}} = \alpha\rho_{\mathcal{S}} + \beta\sigma_{\mathcal{S}}$, for α, β numbers and ρ, σ density matrices.
- Is idempotent: $(\rho_{\mathcal{S}})_{\mathcal{S}} = \rho_{\mathcal{S}}$.
- Is trace preserving: $\mathrm{Tr}\rho_{\mathcal{S}} = \mathrm{Tr}\rho$.

This map has the interpretation of killing the non-symmetric part of a state ρ, just leaving the part that remains invariant under the symmetry. Therefore, $\rho_{\mathcal{S}} = \rho$ if and only if ρ is symmetric. This property makes $S(\rho||\rho_{\mathcal{S}})$ a suitable candidate for studying symmetry breaking. Moreover, if ρ is the density matrix of a vacuum state, the relative entropy becomes a candidate for characterizing spontaneous symmetry breaking, thus making it an entropic order parameter.

For discrete groups this map takes the simple form:

$$\rho_G = \frac{1}{|G|} \sum_{g \in G} \mathcal{L}_g \rho \mathcal{L}_{g^{-1}} \,. \tag{1.121}$$

We may immediately notice that if the state is invariant under the symmetry action (symmetric phase) then $[\mathcal{L}_g, \rho] = 0$, implying that the density matrix is invariant under symmetrization $\rho_G = \rho$.

To illustrate the action of the symmetrizer let us consider a theory with a $\mathbb{Z}_2$ global symmetry. We denote the symmetry lines by $\{\mathbb{1}, \eta\}$. In the symmetric phase, the theory has a unique vacuum state that is invariant under the symmetry, thus it satisfyes $\rho_G = \rho$ and the relative entropy vanishes $S(\rho||\rho_G) = S(\rho||\rho) = 0$. In the SSB phase there are two vacuum states: $|\uparrow\rangle$ and $|\downarrow\rangle$. These vacua are related by the symmetry $\eta\,|\uparrow\rangle = |\downarrow\rangle$. Let us take the density matrix of one of the states $\rho_\uparrow = |\uparrow\rangle\langle\uparrow|$. The action of the symmetrizer (1.121) yields:

$$\rho_{\uparrow,G} = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) \,. \tag{1.122}$$

As they are different vacua, these states are orthogonal $\langle\uparrow \mid \downarrow\rangle = 0$. Thus, the Rényi relative entropies:

$$S^{(n)}(\rho_\uparrow||\rho_{\uparrow,G}) = \frac{1}{n-1}\log\left(2^{n-1}\frac{\mathrm{Tr}[\rho_\uparrow^n]}{\mathrm{Tr}[\rho_\uparrow(\rho_\uparrow + \rho_\downarrow)^{n-1}]}\right) = \log 2 \,. \tag{1.123}$$

Trivially, by taking the limit $n \to 1$, the relative entropy is just

$$S(\rho_\uparrow || \rho_{\uparrow,G}) = \log 2 \neq 0\,, \tag{1.124}$$

and an analogous result is obtained for the other vacuum state $|\downarrow\rangle$. In summary, the relative entropy between the vacuum state and its symmetrized version vanishes in the symmetric phase and gives a result different from zero in the spontaneous symmetry phase. Therefore, relative entropy may identify different spontaneous symmetry breaking patterns, indicating in which phase the theory is, making the quantity a suitable candidate for an entropic order parameter.

In the context of the generalized Landau paradigm, the proposal of relative entropy as an entropic order parameter presents a computationally-accessible way to characterize phases of matter that might be related to generalized symmetries. This approach presents advantages for phases that are not included in the original Landau paradigm, like phases of gauge theories or topological phases, as order parameters may be difficult to identify in those cases. The relative entropy allows us to work directly at the level of vacua structure, circumventing the problem of finding local operators that behave as order parameters.

The proposal of relative entropy as an entropic order parameter has been studied specially in the case of ordinary group-like symmetries [94, 95, 99], where entanglement asymmetry [98] has also been proposed as an order parameter, being both approaches equivalent in some cases. In the context of generalized symmetries there have been some recent extensions to the study of entropic order parameters in the context of categorical symmetries [100, 101, 102, 103, 104], and higher-form symmetries in higher-dimensional theories [105]. In both contexts, entropic order parameters present a way to approach the study of phases of matter in the context of the generalized Landau paradigm.

Finally, in Chapter 5, we will explore the ideas presented above regarding entropic order parameters in theories with categorical symmetries. We will propose a symmetrization map to include these symmetries and compute relative entropy following the ideas exposed above. With this map, we are able to count the number of configurations of line insertions that leave the replica partition function invariant in a CFT. This quantity is believed to be sensitive to a SSB setting as different vacua structure would change then the number of configurations that leave invariant the partition function. The result obtained is only dependent on quantum dimensions, which also suggests the quantity could be sensitive to a SSB setup. With this work, we present a first approach to characterize phases of matter associated to non-invertible symmetries using the relative entropy as an entropic order parameter.

1.4 | Objectives and structure of the thesis

This thesis is structured as a compendium of papers whose purpose is studying symmetries and anomalies from different points of view, focusing mainly on entropic quantities as a way to tackle the problem. Each of the following chapters consist in a paper that has been published during the development of the thesis. In Chapter 2 we studied how to connect the chiral anomaly from the microscopical QFT side to anomalous transport effects by direct integration of the anomalous divergencies, using a geometry that interpolates between the QFT and an ensamble with temperature and chemical potential, that is, a black hole geometry. Chapter 3 presents the first calculation of symmetry resolved entanglement entropy for categorical symmetries, focusing on the resolution of the untwisted sector of 1+1-dimensional rational CFTs. In Chapter 4 we extended the symmetry resolution to anyonic sectors, thus presenting a full resolution with respect to charged sectors given by irreducible representations of the tube algebra. Chapter 5 presents a novel extension of entropic order parameters for categorical symmetries in 1+1-dimensional QFTs, where we proposed a symmetrization map including categorical symmetries and computed relative entropy as an entropic order parameter. Lastly, in chapter 6 I will present a summary of the conclusions and future prospects of the papers composing this text.

Chapter 2

Anomalous transport from geometry

Anomalous transport from geometry

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(Received 17 January 2023; accepted 19 May 2023; published 6 June 2023)

We revisit the relation between black hole geometries and chiral transport. Integrating the anomaly equation in a black hole geometry allows us to derive the chiral transport coefficients for the thermal gas far from the horizon. The key ingredient is to impose vanishing of the covariant current on the horizon. We extend the method to include the usual gauge anomaly for charged black holes and to weak magnetic fields. This allows us to derive the full set of transport coefficients describing the chiral magnetic and chiral vortical effects.

DOI: 10.1103/PhysRevD.107.125003

I. INTRODUCTION

Chiral anomalies are among the most emblematic features of the quantum field theory of relativistic fermions [1]. Gauge symmetries have to be free of anomalies, and this leads to severe constraints on the fermion content of quantum gauge theories. A somewhat different aspect of chiral anomalies is that they also give rise to dissipationless currents when theories containing chiral fermions are put into a finite temperature and finite density (near-)equilibrium state (see [2,3] for reviews).

A subtle issue of chiral anomalies is that they come in consistent and covariant forms [4]. Consistent anomalies are defined via the Wess-Zumino consistency condition, meaning that the anomaly is integrable and stems from a (nonlocal) functional, the 1PI(particle irreducible)-effective action $\Gamma_{1\mathrm{PI}}[A,g]$ in which the gauge field A_μ and the metric $g_{\mu\nu}$ serve as sources for the current J^μ and the energy momentum tensor $T^{\mu\nu}$. Typically the precise form of consistent anomalies depends on a choice of local counterterms that can be added to the 1PI-effective action. A different way of defining anomalies is via their covariant form. There, one demands that the quantum operators J^μ and $T^{\mu\nu}$ are themselves covariant objects under the symmetry transformations (even the anomalous ones). The covariant anomalies in the conservation equations for the currents and energy-momentum tensor are

$$\nabla_\mu J^\mu_a = -\frac{d_{abc}}{32\pi^2}\epsilon^{\mu\nu\sigma\rho}F^b_{\mu\nu}F^c_{\sigma\rho} - \frac{b_a}{768\pi^2}\epsilon^{\mu\nu\sigma\rho}R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\sigma\rho}, \quad (1)$$

$$\nabla_\mu T^{\mu\nu} = F^{a,\nu\mu}J_{a,\mu} - \frac{b_a}{384\pi^2}\epsilon^{\sigma\rho\alpha\beta}\nabla_\mu(F^a_{\sigma\rho}R^{\nu\mu}{}_{\alpha\beta}). \quad (2)$$

We should stress here that the gauge fields A^a_μ as well as the metric $g_{\mu\nu}$ are viewed as classical fields in these equations. The presence of the anomalies is completely determined by the anomaly coefficients $d_{abc} = \sum_r q^r_a q^r_b q^r_c - \sum_l q^l_a q^l_b q^l_c$ and $b_a = \sum_r q^r_a - \sum_l q^l_a$, where $q^{r,l}_a$ is the charge of the right- (left-) handed fermion under the symmetry indexed by a.[1]

It has been noticed a long time ago that chiral fermions at finite density and chemical potential subject to a magnetic field or to rotation give rise to nonvanishing expectation values [5,6]. Furthermore, it turns out that these currents are completely determined by the anomaly coefficients d_{abc} and b_a,

$$\vec{j}_a = \frac{d_{abc}\mu_b}{4\pi^2}\vec{B}_c + \left(\frac{d_{abc}\mu_b\mu_c}{4\pi^2} + \frac{b_a T^2}{12}\right)\vec{\Omega}, \quad (3)$$

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[1]We restrict here to the relevant case of having $U(1)$ symmetries only.

$$\vec{j}_e = \left(\frac{d_{abc}\mu_b\mu_c}{8\pi^2} + \frac{b_a T^2}{24}\right)\vec{B}_a + \left(\frac{d_{abc}\mu_a\mu_b\mu_c}{8\pi^2} + \frac{b_a\mu_a T^2}{24}\right)\vec{\Omega}, \tag{4}$$

where $\vec{B}_a = \vec{\nabla}\times\vec{A}_a$ is a magnetic field of the gauge field coupling to the current J^μ_a and $\vec{\Omega}$ is the vector of angular velocity. These are known as chiral magnetic and chiral vortical effects, respectively.

It has been realized rather early that the terms proportional to only the chemical potentials are determined by the anomaly [7–11]. A typical argument put forward first in [10] relies on constructing an entropy current and demanding that the entropy production rate is positive definite. The temperature dependent parts enter this construction only through undetermined integration constants [11]. The reason for this apparent difference is that the hydrodynamic framework relies on a derivative counting scheme in which the gravitational terms are higher order terms[2] and thus do not appear in Eqs. (3) and (4), which are both of first order in the derivative counting scheme (since $\vec{B}_a = \vec{\nabla}\times\vec{A}_a$ and $\vec{\Omega} = \vec{\nabla}\times\vec{v}$ with $\vec{v}$ the velocity field).

Explicit calculations in field theory and in strongly coupled models based on the gauge duality do, however, indicate that the temperature dependent terms are induced by the gravitational parts of the chiral anomaly [15,16]. In particular, the holographic calculation including the gravitational anomaly is not different from previous ones including only the gauge field anomalies [17,18]. This should be taken as a strong hint that despite the apparent mismatch in derivatives the terms with chemical potential and with temperature follow from the same principles.

It has first been shown that in Euclidean space there is a way of relating the gravitational anomaly terms to the induced current by considering the quantum field theory on a nontrivial geometry that interpolated between empty flat space and finite temperature viewed as the field theory living on $\mathbb{R}^3\times S^1$, where the length of the circle is interpreted as the inverse temperature [19]. Later a Lorentzian construction was put forward in [20]. In Lorentzian terms the geometries that interpolate between flat space and finite temperature can be interpreted as black hole geometries. Indeed, a free falling observer will note nothing special upon crossing the event horizon of a black hole. On the other hand, it is known that a static asymptotic observer at a large distance from the black hole will observe a constant flux of thermal Hawking radiation. Thus, a quantum field theory place in such a black hole background will be in equilibrium with thermal radiation at infinity but experience no temperature at the horizon. It was shown in [20] that by integrating the gravitational parts of the anomaly equations between the horizon and infinity the temperature dependent parts of the currents can be obtained.

This derivation is very different from the entropy considerations based on hydrodynamics. One is then left with the impression that the two contributions to the chiral anomaly are on very different footing. The purpose of this paper is to close a gap in the literature and to show that the argument based on black hole geometries is, in fact, more general and also allows one to recover all the anomaly induced currents that depend on the chemical potential. Thus from this point of view there is really no difference in the two types of anomalies.

The paper is structured as follows. In Sec. II we first review the construction of [20]. This is basically a two-dimensional argument in which the background magnetic field is imagined to be infinitely strong such that the effective dynamics is completely determined by the fermions in the lowest Landau level. In contrast the chiral vortical effect is derived to first order in the vorticity $\vec{\Omega}$. In Sec. III we consider electrically and magnetically charged black holes in $3+1$ dimensions. We show that the temperature and chemical potential dependent terms in the currents can be obtained at once by integrating the anomaly equations. In Sec. IV we show that it is also possible to do this for an electrically charged and rotating black hole, thus allowing us to recover the chiral vortical effect in the energy current. We close in Sec. V with some conclusions and outlook.

II. INTEGRATING THE ANOMALY ON A BLACK HOLE GEOMETRY

From the point of view of quantum field theory the anomalous conservation equations (1) and (2) are operator relations that hold for arbitrary expectation values of matrix elements. On the other hand, the transport phenomena described by (3) and (4) are tied to a particular state in the theory. This is manifest in the explicit expressions for the chiral magnetic effect (CME) and chiral vortical effect (CVE), which are written in terms of state dependent quantities: the temperature T and the chemical potentials μ_a. To establish a precise connection between both perspectives one needs to engineer a background field configuration that mimics appropriately the state, and then integrate the anomalous conservation equation in this background.[3] This is precisely what a generic black hole does. The black hole is generic in the sense that it need not satisfy any equations of motion. Only the generic (topological) properties of the presence of a nondegenerate horizon and an asymptotically flat region connected smoothly are important. As we will see, a chemical

[2]The naive derivative counting argument suggests that the gravitational anomaly gives rise to transport at third order. Such transport coefficients have indeed been discussed recently in [12,13] and earlier in [14].

[3]We point out that the anomalous currents are produced by quantized fields on top of the given background, not by the matter generating the background gravitational field.

potential may be introduced via a "generic" gauge field by considering charged black holes.

We briefly review now the anomaly integration method put forward in [20]; a similar method was originally presented in [21].

A. Thermal CME in 1 + 1 dimensions

First, one considers a system in a strong magnetic field. Fermions will then populate the lowest Landau level, and therefore the relevant physics will be captured by an effective $(1+1)$-dimensional model. In two spacetime dimensions the mixed four-dimensional anomaly for the energy momentum tensor reduces to a pure gravitational anomaly given by

$$\nabla_\mu T^{\mu\nu}_{R,L} = \mp c_g \epsilon^{\nu\sigma}\partial_\sigma R, \tag{5}$$

where R is the Ricci scalar, $\epsilon^{\nu\sigma} = \epsilon(\nu\sigma)/\sqrt{-g}$ is the two-dimensional Levi-Civita tensor, and $\epsilon(\nu\sigma)$ is the totally antisymmetric symbol. For reasons of generality we also allow for a completely general anomaly coefficient c_g. For two dimensional chiral conformal field theories c_g can be identified with the central charge. The subscripts R and L denote right- or left-handed particles, respectively, and the $\pm$ is correlated with the subscript. The central idea of [19,20] is to integrate the anomaly equation (5) on a background that interpolates between the vacuum and the thermal state. With the Lorentzian signature such a background is given by a $(1+1)$-dimensional black hole with Hawking temperature T. The theory at a large distance from the horizon must be in equilibrium with the Hawking radiation emitted by the black hole; thus, the current will be that of a theory at finite temperature with T the Hawking temperature (see also [21]). A generic metric for a two-dimensional black hole is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2. \tag{6}$$

We are considering this metric as a nondynamical metric, so it need not satisfy Einstein's equations. However, we do demand that there is a horizon $f(r_h) = 0$ for some r_h and that the metric is asymptotically flat at infinity $f(r\to\infty) = 1$. The Ricci scalar is given by $R = -f''(r)$, where the prime denotes the derivative with respect to r, so asymptotic flatness further requires $f''(r\to\infty) = 0$. We may also demand that this function goes smoothly to 1 at infinity, that is, $f'(r\to\infty) = 0$. With this metric, we may define the energy current as

$$J^\mu_E = T^{\mu\nu}_{R,L}\eta_\nu, \tag{7}$$

where $\eta = \partial_t$ is a Killing vector field of the chosen metric. The anomaly equation (5) becomes for the energy current

$$\nabla_\mu J^\mu_E = \mp c_g \eta_\nu \epsilon^{\nu\sigma}\partial_\sigma R. \tag{8}$$

We can explicitly compute the right-hand side of the last equation in the black hole background and notice that the term can be written in terms of a total derivative:

$$\nabla_\mu J^\mu_E = c_g \partial_r\left[f(r)f''(r) - \frac{1}{2}f'^2(r)\right]. \tag{9}$$

Thus, we obtain an expression for the energy current that we expect to reproduce the CME. Only radial dependence is assumed. Consequently, we solve for the radial component of the current. As the right-hand side of the equation is a total derivative, the final result is going to depend just on the boundary conditions we have imposed for the function $f(r)$. This is, of course, expected due to the topological nature of anomalies.

We can then arrive to the current of the chiral magnetic effect by integrating the anomaly equation in this background. One should demand that the covariant form of the current vanish at the horizon [20].[4] Equivalently we may argue that the norm of the (covariant) current appearing in (7) should not blow up in any regular point of the spacetime. In particular $J_\mu J^\mu \propto \frac{(J^r)^2}{f(r)}$. This is regular only if the current appearing in (7) vanishes at the horizon. On the other hand, it is at infinity that one observes the thermal radiation, and hence one expects to reproduce the correct result at infinity. Similar arguments apply in subsequent sections. Assuming time independence one finds for the current at infinity

$$J^r_E|_{r=\infty} = 8c_g\pi^2T^2, \tag{10}$$

where T is the Hawking temperature defined by the surface gravity via

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi}. \tag{11}$$

The magnetic field is absent as a consequence of the dimensional reduction. It may be reinstated by noting that the density of states in the lowest Landau level is given by $\frac{B}{2\pi}$. If we also specialize to the case of a single chiral fermion such that $c_g = \pm\frac{1}{96\pi}$ depending on chirality, we get

$$\vec{j}_{E;R,L} = \pm\frac{T^2}{24}\vec{B}. \tag{12}$$

This is indeed the known result for the chiral magnetic effect of a chiral fermion at finite temperature in Eq. (4). Let us comment now on a subtle issue related to the chiral magnetic effect. It is often argued that the CME can be

[4]This is equivalent to the requirement of "consistency with the Euclidean vacuum" introduced in [19,22].

understood easily as a consequence of the physics of the lowest Landau level. Fermions in the lowest Landau level do behave as effective $(1+1)$-dimensional particles, and thus the CME can be viewed as a consequence of the operator relation $J_5^\mu = \epsilon^{\mu\nu} J_\mu$ valid in two spacetime dimensions.[5] This operator relation basically states that the current is given by the axial charge and vice versa. The reduction to $1+1$ dimensions, however, relies on the presence of a strong magnetic field. If this reduction would be essential for the CME, then one should expect corrections to the CME formula for high temperature and small magnetic field. It is, however, well-known that the CME is an exact formula in $3+1$ dimensions and no such corrections arise. Therefore, it is necessary to generalize this argument leading to (12) to four-dimensional black holes and small magnetic fields.

B. Thermal CVE in 3+1 dimensions

This strategy can also be adopted to compute the thermal part of the CVE in $3+1$ dimensions.[6] Inspired by the Kerr metric, the authors of [20] propose to work with the following background metric:

$$ds^2 = -f(z)\frac{(dt-\Omega\rho^2 d\phi)^2}{1-\Omega^2\rho^2} + \frac{dz^2}{f(z)} + d\rho^2 + \frac{\rho^2(d\phi-\Omega dt)^2}{1-\Omega^2\rho^2}, \tag{13}$$

which is in cylindrical coordinates and constructed such that it asymptotes to Minkowski space. Notice that the background encodes rotation, where Ω is the constant angular velocity of the event horizon. Again one imposes the existence of a horizon [$f(z_h)=0$ for some z_h] and asymptotic flatness: $f(z\to\infty)=1$ and $f'(z\to\infty) = f''(z\to\infty)=0$.

Contrary to the previous case, here the first nontrivial contribution arises in the $U(1)$ current J^μ whose anomalous conservation equation in the presence of gravity is proportional to the Pontryagin density:

$$\nabla_\mu J^\mu = -c_g \epsilon^{\mu\nu\rho\sigma} R^\alpha_{\beta\mu\nu} R^\beta_{\alpha\rho\sigma}. \tag{14}$$

It is not hard to show that this equation admits a static solution with z dependence in which only the zth component of the current is nonvanishing. We also keep only the first order in an expansion in the angular velocity Ω since there is no indication that higher order terms are universal. One finds

$$\nabla_\mu J^\mu = -8 c_g f'(z) f''(z)\Omega + \mathcal{O}(\Omega^3). \tag{15}$$

Integrating it and imposing vanishing of the current at the horizon yields to lowest order in Ω,[7]

$$J^z(z\to\infty) = 4 f'^2(z_h)\Omega = 64\pi^2 c_g T^2 \Omega, \tag{16}$$

which for a single chiral fermion gives $\vec{j} = \pm\frac{T^2}{12}\vec{\Omega}$. This coincides with the expression in Eq. (3) specified to a single chiral fermion.

III. CHIRAL MAGNETIC EFFECT IN 3+1 DIMENSIONS VIA ANOMALY

In this section, we are going to present a detailed derivation of the CME for a $U(1)$ current and the energy current using the anomaly integration method explained in the previous section. We consider black hole geometries that interpolate between the vacuum with vanishing currents at the horizon and the state with both chemical potential μ [of the given $U(1)$ symmetry] and temperature T in the presence of a magnetic field B. Thus we consider electrically and magnetically charged black hole geometries [25] hint that a suitable background is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{(2)}, \qquad A = A_t(r)dt - Q\cos\theta d\phi, \tag{17}$$

where Q is the magnetic charge and we are working in spherical coordinates. We fix part of the gauge freedom by using the radial gauge $A_r = 0$. The magnetic field is radial and arises from the ϕ component of the gauge field. By definition the chemical potential is the energy required to thermalize a unit of charge. In the geometry (17) this is the energy needed to bring a unit of charge from infinity to behind the horizon

$$\mu = \int_{r_H}^{\infty} F_{\nu\mu}\eta^\mu dx^\nu, \tag{18}$$

where η^μ is a timelike killing vector and $F_{\mu\nu}$ is the field strength of the gauge field: $F = dA$. Using the Killing vector of the background metric $\eta = \partial_t$ we find $\mu = A_t(\infty) - A_t(r_h)$. As we have restricted ourselves to the

[5]This can also be written as $J^\mu_{R,L} = \pm\epsilon^{\mu\nu} J_{R,L,\mu}$ in terms of right- and left-handed currents.

[6]In this case the dimensional reduction would amount to including a $U(1)$ symmetry associated with the axial symmetry of the system [23].

[7]It is worth mentioning that we obtain a result depending on rotation at infinity although our metric is asymptotically Minkowski, and thus does not rotate at infinity. This is an effect of the coordinates we have chosen. In this specific choice the rotation is at the horizon, but one can perform a change of coordinates to relocate it at infinity, and in that case the result would still remain the same because physics cannot depend on the coordinates one has chosen. In the coordinates in which rotation is present at infinity the chiral vortical effect is better described as a chiral gravito-magnetic effect [24].

radial gauge and only want to consider radial dependence, the residual gauge freedom is simply constant shifts in the gauge field, under which μ is trivially invariant.

We are interested in expressing our result in terms of the actual magnetic field, the one that we can measure. In the coordinates that we are working, the magnetic field does not correspond exactly with the components of the electromagnetic tensor, contrary to what happens in Cartesian coordinates, but they are related by[8]

$$B^i = \frac{1}{2}\frac{\epsilon(ijk)}{\sqrt{-g}}F_{jk}, \tag{19}$$

where latin indices run over spatial coordinates. This gives the promised radial magnetic field $B(r) \equiv B^r(r) = \frac{Q}{r^2}$.

We emphasize again that the background need not satisfy any equations of motion. As part of the background construction one assumes that there is a horizon $f(r_h) = 0$ and asymptotic flatness: $f(r \to \infty) = 1$ and $f'(r \to \infty) = f''(r \to \infty) = 0$.

We can now evaluate the anomaly in the same fashion of the last section. For simplicity of notation we consider a single right-handed fermion. The generalization to a theory with anomaly coefficient d_{abc} is straightforward. We find the following result:

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}J^\mu) = Q\frac{A_t'}{4\pi^2 r^2}. \tag{20}$$

Notice that the Pontryagin density vanishes in this background and only the first piece of the anomaly in (1) contributes. Once we multiply both sides by $\sqrt{-g} = r^2\sin\theta$ the result may be integrated over the region between two concentric spheres, one at the horizon and the other one at infinity. Then, we may use Stokes' theorem, and imposing that the current vanishes on the horizon gives

$$(r^2 J^r)|_\infty = \frac{Q}{4\pi^2}(A_t(\infty) - A_t(r_h)) = \frac{Q}{4\pi^2}\mu. \tag{21}$$

This result is a bit trickier than the one obtained in Sec. II A. If we take a look at the left-hand side of the equation, we may notice that the factor r^2 coming from $\sqrt{-g}$ diverges at infinity. We may momentarily introduce a cutoff scale Λ up to which we integrate in the radial direction to explore the behavior of the current at infinity. Again we impose the boundary condition that the current at the horizon vanishes. Thus, we find

$$J^r(\Lambda) = \frac{1}{4\pi^2}\mu(\Lambda)B^r(\Lambda), \tag{22}$$

where $\mu(\Lambda) = A_t(\Lambda) - A_t(r_h)$. We may notice that this magnetic field decays at infinity, so when we take $\Lambda \to \infty$, the whole term goes to zero. The decay of the field at infinity is indeed a reasonable physical condition. Nonetheless, this does not mean that there is no CME current. To solve the puzzle let us integrate the current over a spherical shell and take the radius to infinity, and that is the flux of the current through a spherical surface. The result is found to be

$$\Phi = \Lambda^2 \int J^r d\Omega^2_{(2)} = \frac{Q}{\pi}\mu, \tag{23}$$

which is obviously nonvanishing. One is thus led to ignore the fact that the magnetic field vanishes at infinity and write the chiral magnetic effect in terms of B_∞, which is to be understood as $B(\Lambda)$ in Eq. (22). With this, we can conclude that the current obtained from the integration of the anomaly

$$\vec{J}_{\rm CME} = \vec{J}(\infty) = \pm\frac{\mu_{R,L}}{4\pi^2}\vec{B}_\infty \tag{24}$$

describes correctly the CME of a chiral $U(1)$ current.

Let us now consider the energy current $J^\mu_E = T^{\mu\nu}\eta_\nu$, with $\eta = \partial_t$ the timelike Killing vector. Symmetry of the energy-momentum tensor and the Killing property of $\nabla_{(\mu}\eta_{\nu)} = 0$ imply

$$(\nabla_\mu T^{\mu\nu})\eta_\nu = \nabla_\mu(T^{\mu\nu}\eta_\nu) - T^{\mu\nu}\nabla_\mu\eta_\nu = \nabla_\mu J^\mu_E. \tag{25}$$

To explicitly evaluate Eq. (2) in the background (17) we first need the current as a function of r, not only the value at infinity that we extracted previously. Integrating Eq. (20) up to a generic point r, it is immediate to find $4\pi^2 J^r = Q(A_t(r) - A_t|_{r_h})/r^2$. Including the latter current into (2) and further contracting with the killing vector gives

$$\nabla_\mu J^\mu_E = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}J^\mu_E) = Q\frac{(A_t - A_t|_{r_h})A_t'}{4\pi^2 r^2} + Q\frac{ff'''}{192\pi^2 r^2}. \tag{26}$$

The first contribution arises from the classical part of the energy-momentum tensor divergence, whereas the second term stems from the anomalous part. Repeating the same integration process as before, now from the horizon up to a generic point r and imposing again that the current vanishes at the horizon, we find

$$(r^2 J^r_E)|_r = Q\frac{(A_t(r) - A_t(r_h))^2}{8\pi^2} + Q\frac{f(r)f''(r) - \frac{1}{2}f'(r)^2}{192\pi^2} + Q\frac{f'(r_h)^2}{348\pi^2}. \tag{27}$$

[8]This is the usual definition of the magnetic field in the hydrodynamics context $B^\rho = \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}u_\lambda F_{\mu\nu}$ once we sit in the local rest frame of the fluid $u_\mu = (1,0,0,0)$.

Evaluating at infinity and using (i) the definition of the chemical potential, (ii) the Hawking temperature, and (iii) the same trick as before to bypass the vanishing magnetic field, we are left with

$$\vec{j}_{E,\mathrm{CME}} = \vec{J}_E(\infty) = \pm\left(\frac{\mu_{R,L}^2}{8\pi^2} + \frac{T^2}{24}\right)\vec{B}_\infty. \tag{28}$$

Also in this case the expression coincides with the relevant term in (4), which contains contributions from both the gauge and the gravitational anomaly as expected.

Notice again that even though the magnetic field decays to 0 at infinity, the flux across a spherical shell at infinity

$$\Phi_E = \int_0^\pi d\theta \int_0^{2\pi} d\phi \Lambda^2 \sin\theta J_E^r = \pm 4\pi Q\left(\frac{\mu_{R,L}^2}{8\pi^2} + \frac{T^2}{24}\right) \tag{29}$$

remains finite.

As a comment we point out that both (24) and (28) are also reproduced by choosing a background gauge field with a uniform magnetic field pointing in the zth direction (strong magnetic field limit). The metric background, though generic, should be consistent with the previous choice, and this would amount to having a metric that breaks the $SO(3)$ rotational symmetry down to $SO(2)$. A good candidate for such a metric is the $\Omega \to 0$ limit of Eq. (13) with possibly a generic function $h(z)$ with the right asymptotics respecting the isometries on the transverse plane:

$$ds^2 = -f(z)dt^2 + \frac{dz^2}{f(z)} + h(z)(d\rho^2 + \rho^2 d\phi^2). \tag{30}$$

It has been written in cylindrical coordinates for convenience. The function $h(z)$, however, does not modify the results provided that we normalize $h(z \to \infty) = 1$. The fact that both a radial decaying magnetic field or an axial constant magnetic field give the right result may be a consequence of the universality of the calculation. However, it is not clear whether one will be preferred over the other if the same method is applied to higher spin currents, for which the procedure is analogous to the number current and the energy momentum tensor.

IV. CHIRAL VORTICAL EFFECT IN 3+1 DIMENSIONS VIA ANOMALY

We extend now the analysis to include the contribution of a $U(1)$ chemical potential in the chiral vortical effect. Though the procedure is by now clear, the choice of a suitable background plays a central role. For the background metric we still use (13):

$$ds^2 = -f(z)\frac{(dt - \Omega\rho^2 d\phi)^2}{1 - \Omega^2\rho^2} + \frac{dz^2}{f(z)} + d\rho^2 + \frac{\rho^2(d\phi - \Omega dt)^2}{1 - \Omega^2\rho^2}. \tag{31}$$

The background comes along with the asymptotics for the generic function. Repeating previous arguments we assume the existence of a horizon at some point z_h as well as asymptotic flatness: $f(z \to \infty) = 1$ and $f'(z \to \infty) = f''(z \to \infty) = 0$. As for the gauge field, recall that the background, though generic, should be able to solve the equations of motion if so required. Taking a look at the field strength of the Kerr-Newmann solution we may infer that a good candidate would be

$$A = A_t(z)dt + \Omega A_\phi(\rho, z)d\phi, \tag{32}$$

where we are again working in the axial gauge $A_z = 0$ and, as explained above, restricting to stationary solutions. Thus, the residual gauge freedom for the time component of the gauge field reduces to constant shifts. However, it is not hard to convince oneself that the ansatz is too generic. By looking at the examples in the previous sections, the only functions that are kept generic encode the information about the state: temperature and chemical potential; yet the ϕ component does not contain such information. Actually, it represents the magnetic field induced by charged particles rotating around the zth axis. We may use this notion to factorize the ρ dependence as $A_\phi = -\rho^2 a_\phi(z)$. Again a_ϕ is too generic. Taking a look at Maxwell's equations of motion suggests that $a_\phi(z) = A_t(z) - G$, with G some integration constant. In particular,

$$\nabla_\mu F^{\mu t} = 0, \tag{33}$$

$$\nabla_\mu F^{\mu\phi} = \Omega\nabla_\mu F^{\mu t} + \Omega(f(A_t' - a_\phi'))' = 0. \tag{34}$$

Consequently, $f(A_t' - a_\phi') = \mathrm{const}$, and evaluating it at the horizon gives $\mathrm{const} = 0$. A final integration gives the sought result $a_\phi(z) = A_t(z) - G$. We will later rederive the same result without making use of the equations of motion.

Notice that a constant shift in A_t may be reabsorbed in a constant shift in G, so that a_ϕ respects the residual gauge invariance. Consequently, we rewrite the generic background as

$$A = (A_t(z) - G)(dt - \rho^2\Omega d\phi), \tag{35}$$

where the constant G has been added also to the temporal component without loss of generality to make manifest that the one-form is proportional to the combination that appears in the metric (31) and hence respects the symmetry

associated with the gravito-magnetic gauge field. In fact, this form of the gauge field is easiest to understand by noting that there is a local velocity field $v_\phi = \rho^2 d\phi$ due to frame dragging. The rotation is then introduced by $dt \to dt - \rho^2\Omega d\phi$ and $d\phi \to d\phi - \Omega dt$. An alternative derivation of (35) consists of requiring that the background respects the gravito-magnetic gauge symmetry $t \to t + \lambda$ and $A_g \to A_g + d\lambda$ where λ is a t independent gauge parameter. This suggests that only the differential one-form $dt - A_g$ appears. Then one notices that rotation to lowest order in Ω is represented by the gravito-magnetic field $A_g = \rho^2\Omega d\phi$. Since we are interested only in the terms linear in Ω, this is sufficient for our purposes.

The magnetic field in the zth direction is obtained using (19): $B^z(z) = 2\Omega(G - A_t(z))$. As boundary conditions for the computation we shall demand that the magnetic field vanishes at infinity. This condition means that there is no additional magnetic field at infinity. We insist that it is imposed in the combination $A_t(z) - G$ and hence respects the residual gauge invariance. Solving the condition for the magnetic field reveals $G = A_t|_\infty$.

Now that the background has been constructed we may proceed to integrate the anomaly equation. Following the steps of [20] presented in Sec. II B we shall assume that only the zth component of the current is needed. In the discussion section we shall comment on the limitations of this assumption. Besides, the computation shall be done on-axis and to leading order in Ω, for higher contributions are not universal as explained above. Substituting the ansatz (31) and (35) into the anomaly equation (1) specified to a single right-handed $U(1)$ current we find

$$\begin{aligned}\nabla_\mu J^\mu &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}J^\mu) = \partial_z J^z \\ &= -\frac{(A_t(z) - A_t|_\infty)A_t'(z)}{2\pi^2}\Omega - \frac{f'(z)f''(z)}{96\pi^2}\Omega + \mathcal{O}(\Omega^3).\end{aligned} \tag{36}$$

Integrating the previous equation gives

$$\begin{aligned}J^z(z) = &-\frac{(A_t(z) - A_t|_\infty)^2}{4\pi^2}\Omega - \frac{f'(z)^2}{192\pi^2}\Omega + \left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\right)\Omega \\ &+ \mathcal{O}(\Omega^3),\end{aligned} \tag{37}$$

where we have written the contribution from the horizon as a function of chemical potential (18) and the Hawking temperature. Evaluating at infinity and keeping the leading piece in Ω we obtain the CVE in the presence of temperature and chemical potential:

$$j_{\mathrm{CVE}} = J^z(\infty) = \left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\right)\Omega. \tag{38}$$

We now proceed with the computation for the energy current which, as before, is defined out of the energy-momentum tensor contracted with a timelike Killing vector (7). Evaluating the anomalous conservation equation (2) in the rotating background (31) and (35) and using Eqs. (25) and (37) we find

$$\begin{aligned}\nabla_\mu J^\mu_E &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}J^\mu_E) = \partial_z J^z_E \\ &= \left(\frac{1}{12}T^2 A_t' + \mu^2\frac{A_t'}{4\pi^2} - \frac{(A_t - A_t|_\infty)^2 A_t'}{4\pi^2}\right)\Omega \\ &\quad - \frac{1}{48\pi^2}\left(\frac{1}{2}A_t'' f f' + A_t'\left[f f'' + \frac{1}{4}f'^2\right]\right. \\ &\quad \left. + \frac{1}{2}(A_t - A_t|_\infty) f f'''\right)\Omega + \mathcal{O}(\Omega^3).\end{aligned} \tag{39}$$

As it is expected, the right-hand side gives a total derivative and the result only depends on the boundary conditions of the background functions. Following the prescription given above, we integrate

$$\begin{aligned}J^z_E(z) = &\left(\frac{1}{12}T^2\bar{A}_t + \frac{\mu^2}{4\pi^2}\bar{A}_t - \frac{\bar{A}_t^3}{12\pi^2}\right)\Omega \\ &- \left(\bar{A}_t' f f' + \bar{A}_t\left[f f'' - \frac{1}{2}f'^2\right]\right)\frac{1}{96\pi^2}\Omega + \mathcal{O}(\Omega^3),\end{aligned} \tag{40}$$

where we have defined $\bar{A}_t = A_t(z) - A_t|_\infty$ for the sake of visual simplicity. Under the same assumptions presented before we recover the CVE for the energy current:

$$j_{E,\mathrm{CVE}} = J^z_E(\infty) = \left(\frac{1}{6}\mu T^2 + \frac{1}{6\pi^2}\mu^3\right)\Omega. \tag{41}$$

Collecting our results for the chiral vortical effects we find

$$\begin{aligned}\vec{j}_{\mathrm{CVE}} &= \left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\right)\vec{\Omega}, \\ \vec{j}_{E,\mathrm{CVE}} &= \left(\frac{1}{6}\mu T^2 + \frac{1}{6\pi^2}\mu^3\right)\vec{\Omega},\end{aligned} \tag{42}$$

which coincides with the relevant terms in Eqs. (3) and (4).

V. SUMMARY AND DISCUSSION

We have revisited the method of integrating the anomaly equation in black hole backgrounds to derive anomaly induced currents at finite temperature and chemical

potential. The new results include the derivation for weak magnetic fields and for the chiral vortical effect in the energy current. The derivations for gauge and gravitational anomalies proceed exactly in the same manner and can be integrated straightforwardly. One obtains the well-known coefficients of chiral magnetic and chiral vortical effects once vanishing of the current at the horizon is imposed as a boundary condition.

We turn our attention now to the assumption the CVE current is parallel to the zth direction. Interestingly, this is not necessarily the case. If we allow in Eq. (37) for a generic current depending on z and ρ,[9] then a new contribution may arise from the ρ component of the current[10]:

$$\nabla_\mu J^\mu = \partial_z J^z(z) + \frac{1}{\rho}\partial_\rho(\rho J^\rho(\rho,z)) = -\frac{(A_t(z)-A_t|_\infty)A_t'(z)}{2\pi^2}\Omega - \frac{f'(z)f''(z)}{96\pi^2}\Omega + \mathcal{O}(\Omega^3). \tag{43}$$

Clearly J^ρ gives a finite contribution so long as $J^\rho = \rho g(z) + \mathcal{O}(\rho^2)$, for an arbitrary function $g(z)$. Thus, the anomaly equation could be integrated with arbitrary contributions from J^ρ and J^z that balance each other. This freedom is not surprising from a mathematical point of view, since the divergence of a vector field does not uniquely determine the vector field. One also needs to know the curl. In two dimensions or in the CME derivation in Sec. III, the amount of symmetry was enough to unequivocally integrate the anomaly equation, but for the CVE in $3+1$ dimensions this is no longer true. Therefore, one needs to include some *ad hoc* knowledge (namely, the current is parallel to the zth direction) to recover the well-known CVE result. Similar considerations apply for the CVE energy current. In this respect we also want to comment on the relation between the chiral magnetic effect in the energy current Eq. (28) and the chiral vortical effect for the current Eq. (38). It follows from basic principles of quantum field theory that these two coefficients are the same. As pointed out some time ago in [26], these transport coefficients can be computed using Kubo formulas in which only roles of the energy and U(1) currents are exchanged. In the particular kinematic limit at zero frequency these have to be the same by complex conjugation. It should be noted that in our calculations the chiral magnetic and chiral vortical coefficients arise from integration in quite different geometries. Using, however, the general fact that the two coefficients have to be the same, this does give another argument why the ρ component of the current in (43) has to vanish once the chiral magnetic effect in the energy current (28) is known.

The method seems to be universally applicable for all theories that feature chiral anomalies. It is, however, well-known that the method fails for spin $3/2$ fields [27]. In this respect it is important to note that the axial current of a relativistic massless spin $3/2$ field is not invariant under the fermionic gauge symmetry and therefore not a good local observable. Taking this into account, it is probably not too surprising that the gravitational anomaly equation does not reproduce the CVE in that case. On the other hand, a new theory for a relativistic spin $3/2$ field with a set of auxiliary fields was introduced in [28]. This theory allows a perturbative coupling to gauge fields and has a gauge invariant axial current. Unfortunately the quantum theory still suffers from pathologies such as a nondefinite Hilbert space [29]. Nevertheless, its anomalous transport properties have recently been studied in a series of papers [30], and the relation to the anomaly integration method has been studied [31].

Similarly there is a gravitational anomaly in spin one gauge fields [32,33] but again the corresponding current (the helicity current) is not a gauge invariant local operator. Interestingly, in that case it has been shown that the integration of the anomaly equation does reproduce the CVE for the helicity current if a mass as regulator of infrared divergencies is introduced [34]. In this respect it would be interesting to study if the anomaly integration method will also fix chiral magnetic and chiral vortical effects in the higher bosonic and fermionic Zilch currents [35–37].

It should also be noted that a global gravitational anomaly has been invoked to fix the temperature dependence of the CVE coefficient in [38,39]. Unfortunately it is so far not known how this argument can be generalized to the spin one case or to the enlarged spin $3/2$ theory.

We also note that upon finishing this manuscript [40] appeared which also studies integration of the anomaly equation in black hole geometries.

ACKNOWLEDGMENTS

This work has been supported by Spanish National Plan Projects PGC2018-095976-B-C21 MCIN, AEI, Fondo Europeo de Desarrollo Regional (FEDER), PGC2018-095862-B-C21 MCIN, AEI, FEDER, Severo Ochoa Center of Excellence CEX2020-001007-S. The work of P. S. B. is supported by Fundación Séneca, Agencia de Ciencia y Tecnología de la Región de Murcia, Grant No. 21609/FPI/21 and Spanish Ministerio de Ciencia e Innovación PID2021-125700NA-C22. S. M. T. is supported by a FPI-UAM predoctoral fellowship.

[9]Dependence on time or on the azimuthal angle breaks the symmetries of the problem.

[10]J^z may also depend on the radial component ρ. Since we are interested in the current on the rotation axis, we could write $J^z(z,\rho) \simeq J^z(z,0) + \mathcal{O}(\rho)$, where the higher order terms do not survive the limit $\rho \to 0$.

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Chapter 3

Categorical-symmetry resolved entanglement in conformal field theory

Categorical-symmetry resolved entanglement in conformal field theory

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(Received 16 February 2024; accepted 19 April 2024; published 23 May 2024)

We propose a symmetry resolution of entanglement for categorical noninvertible symmetries (CaT-SREE) in $(1+1)$-dimensional conformal field theories. The definition parallels that of grouplike invertible symmetries, employing the concept of symmetric boundary states with respect to a categorical symmetry. Our examination extends to rational conformal field theories, where the behavior of CaT-SREE mirrors that of grouplike invertible symmetries. We find that CaT-SREE can be defined if there is no obstruction to gauging the categorical symmetry, as happens in the case of grouplike symmetries. We also provide instances of the breakdown of entanglement equipartition at the next-to-leading order in the cutoff expansion. Our findings shed light on how the interplay between conformal boundary conditions and categorical symmetries lead to specific patterns in the entanglement entropy.

DOI: 10.1103/PhysRevD.109.105026

I. INTRODUCTION

The notion of a global symmetry in quantum field theory (QFT) has been recently generalized in ways that go beyond those described by groups. Central to this is the idea that every symmetry can be associated to a topological operator [1]. The most striking of these generalizations are higher-form symmetries, related to the conservation of extended objects, and categorical or noninvertible symmetries, symmetries whose associated topological operators form a fusion category, that is, do not fuse according to a simple group law. The study of these noninvertible symmetries is providing new and deep insights into the characterization of universal properties of quantum systems of wide interest, spanning condensed matter and high-energy physics (see Refs. [2,3] for a comprehensive and pedagogical review of these developments.).

In $(1+1)$-dimensional conformal field theories (CFTs), on which we focus on this work, noninvertible symmetries implement dualities such as the Kramers-Wannier duality of the $(1+1)$D Ising model [4,5] and the duality between momentum and winding modes (T duality) of the free compactified boson [6–8]. In these theories, a finite categorical symmetry is defined through a fusion category $\mathcal{C}$ of one-dimensional topological defect line operators. In rational conformal field theories (RCFTs) with a diagonal modular invariant partition function [9,10], these topological defect line operators are known as Verlinde lines. Verlinde lines represent both invertible as well as noninvertible symmetries [11,12]. If the set of lines is denoted as $\{\mathcal{L}_i\}_{i\in\mathcal{V}}$ where $\mathcal{V}$ labels the operators $\mathcal{L}_i \in \mathcal{C}$, then the fusion algebra is given by

$$\mathcal{L}_i \times \mathcal{L}_j = \sum_{k\in\mathcal{V}} N_{ij}^k \mathcal{L}_k, \tag{1}$$

where $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ are non-negative integer-valued fusion coefficients. Topological defect lines and particularly Verlinde lines $\mathcal{L}$, do not generically have an inverse $\mathcal{L}^{-1}$ such that $\mathcal{L}\times\mathcal{L}^{-1} = \mathbb{1}$.

Parallel to generalizing the concept of global symmetry, there has been a remarkable interest in understanding the relation between entanglement in QFT and symmetries. In systems with a global grouplike invertible symmetry, this has been carried out through the symmetry resolved entanglement entropy (SREE) [13–15] which intuitively quantifies the amount of entanglement for different charge sectors. Remarkably, it has been shown that at leading order in the UV cutoff expansion, the SRE entropies are equal for all the charge sectors, a result known as entanglement equipartition [15].

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Hitherto, the entanglement behavior in the presence of categorical symmetries in a QFT has been unknown. In this Letter, we establish the SREE for categorical symmetries (CaT-SREE), mirroring the case of grouplike invertible symmetries once the notion of symmetric boundary states with respect to a categorical symmetry is provided [16]. Our results shed light on how certain CFT boundary conditions preserve or enhance certain categorical symmetries, leading to specific patterns in the entanglement entropy. We illustrate the proposal with two RCFTs, the critical Ising model, where it is not possible to obtain a CaT-SREE and the tricritical Ising model, where it is possible, and the result at leading order shows entanglement equipartition.

II. SYMMETRY RESOLVED ENTANGLEMENT IN CFT

In extended quantum systems, the entanglement entropy (EE) measures the amount of quantum correlations between the degrees of freedom located within an arbitrary region A and those sited on its complement B. Assuming that the Hilbert space $\mathcal{H}$ of the system factorizes as $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, where $\mathcal{H}_A$ contains the degrees of freedom in the region A and $\mathcal{H}_B$ the ones in B, for given a pure state $|\Psi\rangle\in\mathcal{H}$, the reduced density matrix of A is defined by tracing out the degrees of freedom corresponding to the complementary region B as $\rho_A=\mathrm{Tr}_{\mathcal{H}_B}|\Psi\rangle\langle\Psi|$.

The entanglement between A and B is thus quantified through the Rényi and entanglement entropies

$$S_A^n=\frac{1}{1-n}\log\mathrm{Tr}\rho_A^n,\qquad S_A=\lim_{n\to1}S_A^n=-\mathrm{Tr}\rho_A\log\rho_A. \tag{2}$$

We consider now there is a local charge operator $\mathcal{Q}=\mathcal{Q}_A\otimes\mathbb{1}_B+\mathbb{1}_A\otimes\mathcal{Q}_B$ that generates a global Abelian symmetry group G in our theory. When $|\Psi\rangle$ is an eigenstate of $\mathcal{Q}$, then $[\rho_A,\mathcal{Q}_A]=0$ and ρ_A is block diagonal $\rho_A=\oplus_Q\,\Pi_Q\rho_A=\oplus_Q\,p_A[Q]\rho_A[Q]$, with $\sum_Q p_A[Q]=1$ and $\mathrm{Tr}\rho_A[Q]=1$, each block corresponding to a charge sector of $\mathcal{Q}_A$ where Q are eigenvalues of $\mathcal{Q}_A$, Π_Q is a projector to the eigenspace of Q and $p_A[Q]=\mathrm{Tr}[\Pi_Q\rho_A]$ is the probability of measuring the charge value Q in the region A. G being Abelian, the eigenvalues Q label the irreducible representations r of the group.

As a result, the entanglement between regions A and B, may be decomposed into the contributions of each charge sector [13–15] through the symmetry resolved Rényi entropy

$$S_A^n[Q]=\frac{1}{1-n}\log\mathrm{Tr}\rho_A^n[Q]. \tag{3}$$

Entanglement equipartition is the situation for which $\mathrm{Tr}\rho_A^n[Q]$ and thus $S_A[Q]$ do not depend on Q. With this, the fundamental object to compute the SREE is the replica partition function [17,18] at a fixed value of charge Q

$$Z_n[Q]=\mathrm{Tr}\Pi_Q\rho_A^n, \tag{4}$$

from which the SREE can be written as

$$S_A^n[Q]=\frac{1}{1-n}\log\frac{Z_n[Q]}{Z[Q]^n},\qquad Z[Q]\equiv Z_1[Q]. \tag{5}$$

In a $(1+1)$-dimensional CFT, the factorization of the Hilbert space $\mathcal{H}$ as $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, requires imposing boundary conditions a, b that preserve conformal symmetry at the entangling surface ∂A. These boundary conditions have nontrivial consequences for the EE [19,20]. Specifying the region A to an interval of length ℓ, this is implemented by encircling the two entangling points at ∂A with two disks of radius $\varepsilon\ll1$, acting as UV cutoffs at which the boundary conditions a and b are imposed (Fig. 1, upper panel). This manifold is mapped into an annulus of length $W=2\log(\ell/\varepsilon)+\mathcal{O}(\varepsilon)$ and circumference 2π ($2\pi n$, after replicating) by a conformal transformation (Fig. 1, lower panel) where the space-time is periodic in one direction and the $|a\rangle$ and $|b\rangle$ states are defined at the ε boundaries. In this geometry, traces of ρ_A^n are evaluated in terms of BCFT partition functions as [21,22]

$$Z_n[q^n]=\mathrm{Tr}_{ab}[\rho_A^n]=\frac{Z_{ab}[q^n]}{Z_{ab}[q]^n},\qquad Z_{ab}[q]=\mathrm{Tr}_{ab}\rho_A=\mathrm{Tr}_{ab}[q^{(L_0-c/24)}], \tag{6}$$

with the Virasoro zero mode L_0 and the central charge c. Here, $\mathrm{Tr}_{ab}\equiv\mathrm{Tr}_{\mathcal{H}_{A,ab}}$ refers to a trace taking into account the nontrivial boundary conditions and $q=e^{2\pi i\tau}$ is the nome with the modular parameter $\tau=i\pi/W$. Therefore, $q=e^{-2\pi^2/W}$ and $\tilde{q}=e^{-2W}$, with $\tilde{q}$ obtained after a modular S transformation, $S\colon\ \tau\to\tilde{\tau}=-1/\tau$.

After imposing the Hilbert space decomposition $\mathcal{H}=\mathcal{H}_{A,ab}\otimes\mathcal{H}_{B,ba}$, the remaining symmetry algebra in a CFT with a global symmetry is called $\mathcal{A}$ and

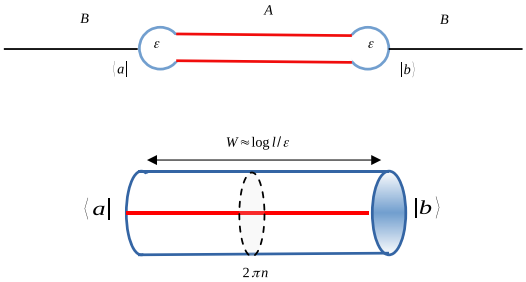


FIG. 1. The factorization ab imposes disks $\varepsilon\ll1$ with boundary conditions a, b (upper panel). The resulting manifold is replicated and after tracing over $\mathcal{H}_{B,ba}$, a conformal transformation yields an annulus of width W and circumference $2\pi n$.

$\mathcal{H}_{A,ab} = \bigoplus_i \mathcal{H}_i^{n_{ab}^i}$ with i running over the allowed representations of $\mathcal{A}$ and the multiplicities n_{ab}^i depending on the boundary conditions a and b. Then, the replica BCFT partition functions can be written in terms of the characters $\chi_i(q) = \mathrm{Tr}_{\mathcal{H}_i}[q^{(L_0-c/24)}]$ for the representation i,

$$Z_{ab}[q^n] = \sum_i n_{ab}^i \chi_i(q^n) = \langle a|\tilde{q}^{\frac{1}{n}(L_0-c/24)}|b\rangle. \tag{7}$$

The last equality is obtained after a modular transformation to the S-dual channel where the boundary condition dependence explicitly appears in terms of Cardy conformal boundary states

$$|a\rangle = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}}|j\rangle\!\rangle, \tag{8}$$

with $|j\rangle\!\rangle$ being an Ishibashi state for the jth representation of $\mathcal{A}$ [23], and coefficients S_{aj} are elements of the modular matrix $\mathcal{S}$ of the CFT [21,24].

III. SREE FOR GROUP SYMMETRIES

The symmetry resolution of entanglement entropy of Abelian group symmetries has been well studied previously [25–28]. The projectors into different irreducible representations of a finite group G are given by

$$\Pi^r = \frac{d_r}{|G|}\sum_{g\in G}\chi_r^*(g)\,\widehat{\mathcal{L}}_g = \frac{d_r}{|G|}\sum_{g\in G}\chi_r^*(g)\xrightarrow{\widehat{\mathcal{L}}_g}, \tag{9}$$

where r labels the irreps of G and thus the different Q-charge sectors, d_r is the dimension of the irrep, $|G|$ is the order of the group, $\chi_r^*(g)$ is the character of the element of $g\in G$ in the irrep r and $\mathcal{L}_g$ is the topological operator implementing the action of g on states supported on the region A.

Using projectors (9) one may write the partition function associated to a charge sector labeled by r in Eq. (4) as

$$Z_{ab}[q^n, r] = \mathrm{Tr}_{ab}\Pi^r\rho_A^n = \frac{d_r}{|G|}\sum_{g\in G}\chi_r^*(g)\frac{Z_{ab}[q^n,g]}{Z_{ab}^n[q]},$$
$$Z_{ab}[q^n, g] = \mathrm{Tr}[\hat{\mathcal{L}}_g q^{n(L_0-c/24)}], \tag{10}$$

where the explicit action of the topological operator $\hat{\mathcal{L}}_g$ is encoded in the charged moment $Z_{ab}[q^n, g]$ (one for each element of the group). Here, we use $\mathcal{L}_g$ for a topological line in Euclidean spacetime, and $\hat{\mathcal{L}}_g$ for the corresponding operator acting on the Hilbert space. As before, one may express $Z_{ab}[q^n, g]$ in the S-dual channel in terms of boundary states $|a\rangle$ and $|b\rangle$ as

$$Z_{ab}[q^n, g] =_g \langle a|\tilde{q}^{\frac{1}{n}(L_0-c/24)}|b\rangle_g, \tag{11}$$

where the subindex g represents that the states belong to the Hilbert space generated by inserting the operator $\hat{\mathcal{L}}_g$ as a defect operator in the original theory, which is the defect or twisted Hilbert space $\mathcal{H}_{\mathcal{L}_g}$. Thus, in this approach, computing SREE reduces to find suitable boundary states $|a\rangle_g$ and $|b\rangle_g$. Namely, as $Z_{ab}[q^n, g]$ is defined through the insertion of $\mathcal{L}_g$ in the annulus partition function, it is required that $\mathcal{L}_g$ can end topologically on the boundary of the interval which imposes a constraint on the allowed boundary states in the dual S channel [16].

For invertible group symmetries the topological endability is equivalent to having G invariant boundary states. A natural definition in the S-dual channel for a (conformal) boundary a to be G symmetric is

$$\hat{\mathcal{L}}_h|a\rangle_g = |a\rangle_g, \quad \forall\ h\in G. \tag{12}$$

For finite groups the result at leading order in the limit when $\varepsilon \ll \ell$ (where $q\to 1$ and $\tilde{q}\to 0$) is quite simple. There, the main contribution comes from the untwisted sector [29], that is to say, the vacuum state propagation is the major contribution to the amplitude in the S-dual channel and the SREE reads as [25–28]

$$S_A[q, r] = \frac{c}{3}\log\frac{\ell}{\varepsilon} + \log\frac{d_r^2}{|G|} + \mathrm{g}_a + \mathrm{g}_b, \tag{13}$$

where $\mathrm{g}_a = \log\langle 0|a\rangle$, $\mathrm{g}_b = \log\langle 0|b\rangle$ are the Affleck-Ludwig boundary entropies [30], and d_r is the dimension of the irrep r. The term $\mathcal{O}(\log\ell/\varepsilon)$ captures the equipartition of EE among distinct charge sectors, primarily at the leading order. This equal distribution is broken by the term of order $\mathcal{O}((\ell/\varepsilon)^0)$ by the negative term $\log p_r$, with $p_r = d_r^2/|G|$ representing the probability of measuring the representation r within block A, a scenario denoted as weak entanglement equipartition to distinguish it from the strong equipartition. A parallel outcome was observed in the examination of Wess-Zumino-Witten models [31].

IV. CATEGORICAL-SYMMETRY RESOLVED ENTANGLEMENT ENTROPY

We propose the symmetry resolution of entanglement for CaT-SREE in analogy with the BCFT approach for grouplike invertible symmetries. For this, it is necessary to define topological endability and thus, symmetric boundary conditions, for the case of (categorical) noninvertible symmetries. These have been proposed in [16] through the notions of strongly symmetric and weakly symmetric boundary states. While these two concepts are equivalent for invertible grouplike symmetries, they diverge for categorylike noninvertible symmetries.

Recalling the fusion algebra in Eq. (1), we focus on the finite subset of boundary conditions $\{a\}_{a\in\mathcal{B}}$ with $\mathcal{B}$ labeling these boundaries, related by the action of a finite symmetry fusion category $\mathcal{C}$. The corresponding boundary states are denoted as $\{|a\rangle\}$. This is known as a module category and the action of $\mathcal{L}_i$ acting on such a class of boundary a is given by

$$\mathcal{L}_i \otimes a = \bigoplus_{b\in\mathcal{B}} \tilde{N}_{ia}^{b} b, \tag{14}$$

where $\tilde{N}_{ia}^{b} \in \mathbb{Z}_{\geq 0}$. With this, two notions of $\mathcal{C}$-symmetric boundary states can be established [16]: A conformal boundary condition a is $\mathcal{C}$-strongly symmetric if the corresponding boundary state $|a\rangle$ is an eigenstate under the action of $\mathcal{C}$ with eigenvalues given by the quantum dimensions $\langle\mathcal{L}_i\rangle$,

$$\hat{\mathcal{L}}_i|a\rangle = \langle\mathcal{L}_i\rangle|a\rangle \quad \forall\ \mathcal{L}_i \in \mathcal{C}. \tag{15}$$

This definition reduces to a G-symmetric boundary condition in the case of grouplike invertible symmetries. On the other hand it is considered that a conformal boundary condition a is $\mathcal{C}$-weakly symmetric if every topological line in $\mathcal{C}$ can end topologically on a. Operationally speaking this means that $\tilde{N}_{ia}^{a} \geq 1$ for every $\mathcal{L}_i$ in $\mathcal{C}$, which implies

$$\hat{\mathcal{L}}_i|a\rangle = |a\rangle \oplus \cdots \quad \forall\ \mathcal{L}_i \in \mathcal{C}. \tag{16}$$

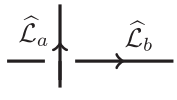
This second notion of $\mathcal{C}$-symmetric boundary condition relax enough the requirements for finding the appropriate boundary states needed to define SREE for fusion categorical noninvertible symmetries.

A. CaT-SREE in RCFT

The simplest models to define the SREE for fusion CaT-SREE are two-dimensional RCFTs, for which there exists a correspondence between Verlinde lines and bulk primary operators [32]. Thus, each line representing a symmetry of the model is associated with one primary operator, and their fusion rules are those given by the operator product expansion coefficients of the corresponding primaries.

The first step to define CaT-SREE is to write a full set of projectors associated to the elements of fusion category $\mathcal{C}$ in terms of elements of the modular matrix $\mathcal{S}$ of the CFT [33]:

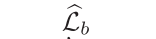
$$\Pi_a{}^{c} = \sum_b S_{0c}\bar{S}_{bc} \tag{17}$$



where $\{\hat{\mathcal{L}}_a\}_{a\in\mathcal{V}} \in \mathcal{C}$. The lines pictorially represent the Verlinde lines of the RCFT inserted either along the time direction in the annulus (vertical ones), which twist the Hilbert space of the theory, or along the spatial direction, which amounts to charged operators acting over the states on the Hilbert space (horizontal ones) [11].

As we are interested in resolving EE on the original Hilbert space of the theory, we will consider only projectors of the form $\Pi_{\mathbb{1}}^{c}$. Here, we write these projectors in full analogy with the grouplike symmetry case Eq. (9) as

$$\Pi_{\mathbb{1}}{}^{c} := \Pi^{c} = \frac{d_c}{|\mathcal{C}|}\sum_{b\in\mathcal{C}} \chi_c^*(b) \tag{18}$$



by defining $d_c = \frac{S_{0c}}{S_{00}}$ as the quantum dimension of the line $\hat{\mathcal{L}}_c$, the order of the category $|\mathcal{C}| = \sum_c d_c^2$, and the characters $\chi_c^*(b) = \frac{\bar{S}_{bc}}{S_{00}}$.

We note that the projectors in Eq. (18) are written for a simple element of the category $\mathcal{L}_c \in \mathcal{C}$. However, the element labeling an irrep of the category is not, in general, a simple object and may be described by nonsimple topological lines whose associated projectors can be written as (18) [34,35].

Thus, in analogy with grouplike invertible symmetries, we define the CaT-SREE in terms of the partition functions

$$Z_{c_1c_2}[q^n, a] = \mathrm{Tr}_{c_1c_2}[\Pi^a \rho_A^n] = \frac{d_a}{|\mathcal{C}|}\sum_{b\in\mathcal{C}} \chi_a^*(b)\frac{Z_{c_1c_2}[q^n, b]}{(Z_{c_1c_2}[q])^n}, \tag{19}$$

where the generalized charged moment in the S-dual channel is defined as

$$Z_{c_1c_2}[q^n, b] =_b \langle c_1|\tilde{q}^{\frac{1}{n}(L_0 - c/24)}|c_2\rangle_b, \tag{20}$$

and $|c_{1,2}\rangle_b$ are Cardy boundary states in the $\mathcal{C}$-weakly symmetric sense exposed above.

We illustrate our definition with two examples, the critical Ising model and the tricritical Ising model.

V. THE CRITICAL ISING MODEL

The critical Ising model is described by a $(1+1)$-dimensional RCFT with a central charge $c = \frac{1}{2}$. There are three primary operators in the model: the identity $\mathbb{1}$, the energy field ϵ, and the spin field σ. The symmetries of this model are described by three Verlinde lines: $\{\hat{\mathbb{1}}, \hat{\eta}\}$ which conform the usual $\mathbb{Z}_2$ symmetry of the Ising model, and $\hat{\mathcal{N}}$ that implements the Kramers-Wannier duality [4,5]. These lines follow the fusion rules of the Ising category

$$\eta \times \eta = \mathbb{1}, \qquad \mathcal{N} \times \mathcal{N} = \mathbb{1} + \eta, \qquad \eta \times \mathcal{N} = \mathcal{N} \tag{21}$$

As discussed above, to obtain the CaT-SREE one must first compute the set of the $\mathcal{C}_{\mathrm{Ising}}$-symmetric Cardy states through (8). In doing so, it is noticed that there are three simple boundary states in this model,

$$|\mathbb{1}\rangle = \frac{1}{\sqrt{2}}|\mathbb{1}\rangle\!\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle\!\rangle + \frac{1}{2^{1/4}}|\sigma\rangle\!\rangle,$$
$$|\epsilon\rangle = \frac{1}{\sqrt{2}}|\mathbb{1}\rangle\!\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle\!\rangle - \frac{1}{2^{1/4}}|\sigma\rangle\!\rangle,$$
$$|\sigma\rangle = |\mathbb{1}\rangle\!\rangle - |\epsilon\rangle\!\rangle. \tag{22}$$

The boundary states $|\mathbb{1}\rangle$, $|\epsilon\rangle$, and $|\sigma\rangle$ conform the Ising, or more technically, the Tambara-Yamagami $\mathrm{TY}_+\mathbb{Z}_2$ fusion category, that is a regular module category. Therefore, there is a one-to-one correspondence between the boundary states (22) and the Verlinde lines such that $|\mathbb{1}\rangle \equiv \hat{\mathbb{1}}, |\epsilon\rangle \equiv \hat{\eta}, |\sigma\rangle \equiv \hat{\mathcal{N}}$ with

$$\hat{\eta}|\mathbb{1}\rangle = |\epsilon\rangle, \quad \hat{\eta}|\epsilon\rangle = |\mathbb{1}\rangle, \quad \hat{\eta}|\sigma\rangle = |\sigma\rangle,$$
$$\hat{\mathcal{N}}|\mathbb{1}\rangle = |\sigma\rangle, \quad \hat{\mathcal{N}}|\epsilon\rangle = |\sigma\rangle, \quad \hat{\mathcal{N}}|\sigma\rangle = |\mathbb{1}\rangle \oplus |\epsilon\rangle. \tag{23}$$

We note that only $|\sigma\rangle$ is invariant under the action of the group $\mathbb{Z}_2$. In this sense, $|\sigma\rangle$ is a $\mathbb{Z}_2$-symmetric state, and thus it is possible to use it to compute the SREE for the $\mathbb{Z}_2$ grouplike symmetry of the model [27]. However, none of the three boundary states are symmetric, neither in the strong nor weak sense, under the action of $\mathcal{N}$. As a result, it is not possible to define the CaT-SREE in the critical Ising model. A fusion category can only admit a strongly symmetric boundary if it is anomaly free, while it admits a weakly symmetric boundary if and only if it can be "gauged" in the generalized sense posed in [16]. Being $\mathcal{C}_{\mathrm{Ising}}$ anomalous, hence the impossibility of defining the CaT-SREE from the obstruction to gauging it, as just happens in the case of group-like symmetries [27,36].

VI. THE TRICRITICAL ISING MODEL

The tricritical Ising model is a RCFT with central charge $c = \frac{7}{10}$. The model is composed by six primary operators and six lines. In addition to the trivial line $\mathbb{1}$ and the $\mathbb{Z}_2$ invertible line η, there are four more simple lines, W, ηW, $\mathcal{N}$, and $W\mathcal{N}$. Nontrivial fusion rules for these lines are given by

$$\eta \times \eta = \mathbb{1}, \qquad \mathcal{N} \times \mathcal{N} = \mathbb{1} + \eta,$$
$$\eta \times \mathcal{N} = \mathcal{N} \times \eta = \mathcal{N}, \qquad W \times W = \mathbb{1} + W. \tag{24}$$

From these relations we can identify $\{\mathbb{1}, \eta, \mathcal{N}\}$ as a $\mathrm{TY}_+\mathbb{Z}_2$ subcategory and $\{\mathbb{1}, W\}$ a Fibonacci subcategory $\mathcal{C}_{\mathrm{Fib}}$. Same as with critical Ising model, for the first group of lines we cannot find symmetric boundary conditions, neither strong nor weak. However, that is not the case for $\mathcal{C}_{\mathrm{Fib}}$ that is the simplest example of a category that can be gauged, and therefore admits a weakly symmetric boundary. Namely, there are three boundary states that are weakly symmetric under $\mathcal{C}_{\mathrm{Fib}}$:

$$\hat{W}|W\rangle = |W\rangle \oplus |\mathbb{1}\rangle, \qquad \hat{W}|\eta W\rangle = |\eta W\rangle \oplus |\eta\rangle,$$
$$\hat{W}|W\mathcal{N}\rangle = |W\mathcal{N}\rangle \oplus |\mathcal{N}\rangle. \tag{25}$$

For simplicity, we choose to work with the boundary condition $|W\mathcal{N}\rangle$. Through Eq. (8), this state can be written as

$$|W\mathcal{N}\rangle = \frac{1}{\sqrt{N}}\left(|\mathbb{1}\rangle\!\rangle + \varphi^{-3/2}|\epsilon\rangle\!\rangle - \varphi^{-3/2}|\epsilon'\rangle\!\rangle - |\epsilon''\rangle\!\rangle\right), \tag{26}$$

with $\varphi = \frac{1+\sqrt{5}}{2}$ the golden ratio and $N = (\frac{10}{5+2\sqrt{5}})^{1/2}$. Thus, the charged moment associated to the untwisted sector is given in terms of the Virasoro characters by (see Supplemental Material [37]):

$$Z_{W\mathcal{N}}[q^n, \mathbb{1}] = \frac{1}{N}\Big[\chi_0\big(\tilde{q}^{\frac{1}{n}}\big) + \varphi^{-3}\chi_{\frac{1}{10}}\big(\tilde{q}^{\frac{1}{n}}\big) + \varphi^{-3}\chi_{\frac{3}{5}}\big(\tilde{q}^{\frac{1}{n}}\big) + \chi_{\frac{3}{2}}\big(\tilde{q}^{\frac{1}{n}}\big)\Big], \tag{27}$$

where, for notational convenience, we use $Z_{aa} \to Z_a$.

In order to compute the charged moment $Z_{W\mathcal{N}}[q^n, W]$, we need an explicit expression of the boundary state $|W\mathcal{N}\rangle$ twisted by the introduction of the Verlinde line $\hat{W}$ as a defect operator. In general, this new boundary state is given by a combination of twisted Ishibashi states, that is, conformal scalars on the W-twisted Hilbert space. The twisted W-Hilbert space contains nine primary operators; among them there are three scalars, ϵ_W, ϵ'_W, σ_W with conformal weights $\frac{1}{10}$, $\frac{3}{5}$, and $\frac{3}{80}$, respectively. This implies that the $\mathcal{C}_{\mathrm{Fib}}$-symmetric twisted Cardy state is a linear combination of the twisted Ishibashi states associated with these operators,

$$|W\mathcal{N}\rangle_W = \alpha_1|\epsilon\rangle\!\rangle_W + \alpha_2|\epsilon'\rangle\!\rangle_W + \alpha_3|\sigma\rangle\!\rangle_W, \tag{28}$$

for some fixed coefficients α_i. With this state, the twisted charged moment is

$$Z_{W\mathcal{N}}[q^n, W] = \alpha_1^2\chi_{\frac{1}{10}}\big(\tilde{q}^{\frac{1}{n}}\big) + \alpha_2^2\chi_{\frac{3}{5}}\big(\tilde{q}^{\frac{1}{n}}\big) + \alpha_3^2\chi_{\frac{3}{80}}\big(\tilde{q}^{\frac{1}{n}}\big). \tag{29}$$

Comparing the two contributions for the CaT-SREE of the $\mathcal{C}_{\mathrm{Fib}}$ subcategory of this model, one notices that $Z_{W\mathcal{N}}[q^n, \mathbb{1}]$ dominates over $Z_{W\mathcal{N}}[q^n, W]$ at leading order in the ϵ expansion (see the Supplemental Material [37]).

In order to find the CaT-SREE, we note that there are 2 irreps of $\mathcal{C}_{\mathrm{Fib}}$ that we label with $\mathrm{r}_{\mathcal{C}} = \{A, B\}$. With this, one may write the projectors associated to those representations as a combination of projectors of the form (18),

$$\Pi^A = \Pi^{\mathbb{1}} + \Pi^{\eta} + \Pi^{\mathcal{N}},$$
$$\Pi^B = \Pi^{W} + \Pi^{\eta W} + \Pi^{W\mathcal{N}}, \tag{30}$$

which explicitly read as

$$\Pi^{\mathrm{r}_\mathcal{C}} = \frac{d_{\mathrm{r}_\mathcal{C}}}{|\mathcal{C}|}\left(d_{\mathrm{r}_\mathcal{C}} \xrightarrow{\hat{\mathbb{1}}} + \chi^*_{\mathrm{r}_\mathcal{C}}(W) \xrightarrow{\widehat{W}} \right), \tag{31}$$

where $d_A = 1$ and $d_B = \varphi$ are the quantum dimensions of the Fibonacci anyons, the total quantum dimension is given by the usual formula $|\mathcal{C}| = d_A^2 + d_B^2 = 1 + \varphi^2$, and the characters are $\chi_A^*(W) = \varphi$ and $\chi_B^*(W) = -1$.

These projectors coincide with those characterizing a theory with Fibonacci anyons, those describing the low-energy physics of the fractional quantum Hall effect at filling factor $\nu = 5/2$ [38].

As there are two simple anyons in this category, there are two projectors of the type of Eq. (18). This ensures that the projectors (31) successfully project into the irreps of the Fibonacci subcategory $\mathcal{C}_{\mathrm{Fib}}$.

With this, the partition functions associated to each charged sector (10) are given by

$$Z[q^n, \mathrm{r}_\mathcal{C}] = \frac{d_{\mathrm{r}_\mathcal{C}}}{|\mathcal{C}|}\left(d_{\mathrm{r}_\mathcal{C}} \frac{Z[q^n, \mathbb{1}]}{(Z^n[q]} + \chi^*_{\mathrm{r}_\mathcal{C}}(W) \frac{Z[q^n, W]}{Z^n[q])} \right), \tag{32}$$

where the subscript $W\mathcal{N}$ has been suppressed for convenience. From these, the CaT-SREE at leading order for both irreps $\mathrm{r}_\mathcal{C} = \{A, B\}$ reads as

$$S[q, \mathrm{r}_\mathcal{C}] = \frac{c}{3}\log\frac{\ell}{\varepsilon} + \log\frac{d^2_{\mathrm{r}_\mathcal{C}}}{|\mathcal{C}|} + 2\mathrm{g}_{W\mathcal{N}}, \tag{33}$$

with $\mathrm{g}_{W\mathcal{N}} = \log\langle\langle \mathbb{1}|W\mathcal{N}\rangle$ the Affleck-Ludwig boundary entropy. This is the main result in this work. Formally, it is analogous to the one obtained for finite groups (13), that is, the CaT-SREE at leading order in the UV cutoff is equally distributed among the different (Fibonacci anyon) charge sectors A and B. Similarly to invertible grouplike symmetries, the entanglement equipartition is broken by constant terms related to the quantum dimension $d_{\mathrm{r}_\mathcal{C}}$ and the boundary entropy.

We note that the same entanglement resolution can be obtained for the tetracritical Ising model, whose topological lines generate the nonanomalous symmetry $\mathrm{Rep}(S_3)$ that possesses weakly symmetric boundary states under the subcategory $\mathcal{C}_{\mathrm{Fib}}$. Finally, we remark that, in some cases, the CaT-SREE can be obtained in terms of strongly symmetric states, as, for instance, the double Ising CFT, which admits a strongly symmetric boundary condition with respect to the categorical symmetry $\mathrm{Rep}(H_8)$ [16].

VII. CONCLUSIONS

We have shown how categorical symmetries shape the entanglement structure in CFTs by proposing a symmetry resolution of entanglement for these symmetries. Our proposal has then provided new insights into the relationship between EE and boundary conditions. As the EE of a topological phase reflects the fusion rules of anyons and their braiding statistics, a deeper understanding of the imprint of categorical symmetries on entanglement provides a future route to characterize topological order. It is also interesting to understand in future works, the microscopic description of CaT-SREE by exploring the entanglement properties of noninvertible operators on the lattice [39–42].

ACKNOWLEDGMENTS

We thank Nabil Iqbal, Sridip Pal, Thomas Bartsch, Matthew Bullimore, Karan Bhatia, Tudor Dimofte, Sašo Grozdanov, Giorgio Frangi, Joan Simon Soler, Naveen Balaji Umasankar, and Mile Vrbica for insightful discussions on this project. The work of P. S.-B. is supported by Fundación Séneca de la Región de Murcia, Grant No. 21609/FPI/21. A. D. is supported by the STFC Consolidated Grant No. ST/T000600/1—"Particle Theory at the Higgs Centre." J. M.-V. thanks Programa Recualificación del Sistema Universitario Español 2021–2023. P. S.-B. and J. M.-V. thank the financial support of Spanish Ministerio de Ciencia e Innovación PID2021-125700NAC22. G.-S. acknowledges funding from Grant No. PID2021-127726NB-I00.

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Chapter 4

Generalized symmetry resolution of entanglement in conformal field theory for twisted and anyonic sectors

Generalized symmetry resolution of entanglement in conformal field theory for twisted and anyonic sectors

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(Received 10 September 2024; accepted 15 October 2024; published 2 December 2024)

A comprehensive symmetry resolution of the entanglement entropy in $(1+1)$-d rational conformal field theories with categorical noninvertible symmetries is presented. This amounts to symmetry resolving the entanglement with respect to the generalized twisted and anyonic charged sectors of the theory. The anyonic sectors label the irreducible representations of a modular fusion category defining the symmetry and can be understood through the $(2+1)$-d symmetry topological field theory that encodes the symmetry features of the conformal field theory. Using this, we define the corresponding generalized boundary-dependent charged moments necessary for the symmetry resolution of the entanglement entropy and find that, contrary to the case of invertible symmetries, the entanglement equipartition between different charged sectors at the next-to-leading order in the ultraviolet cutoff, is broken.

DOI: 10.1103/PhysRevD.110.125005

I. INTRODUCTION

Symmetry-resolved entanglement entropy (SREE) is an entanglement measure, that, in extended quantum systems such as quantum field theories (QFTs) with an internal global symmetry, quantifies the amount of entanglement entropy (EE) corresponding to different symmetry sectors. Its study has fostered interest in how global symmetries shape the entanglement structure of these theories (see [1] for a short review on the topic).

Namely, the concept of a global symmetry in QFT has been recently generalized with respect to the standard ways given by group theory. In the new paradigm, one realizes that any symmetry operation can be associated with a topological operator [2]. These generalizations correspond to *higher-form* symmetries, related to the conservation of extended objects, and *categorical* or noninvertible symmetries, for which, the composition law of their topological operators is given by a *fusion category*, in contrast to the simple group multiplication law related to the standard group-like *invertible* symmetries (a pedagogical review can be found in [3]).

The aim of this work is to obtain a full symmetry resolution of the entanglement entropy for $(1+1)$-d rational conformal field theory (RCFT) with a categorical noninvertible symmetry, thus generalizing the proposal in [4]. Specifically, a remarkable relation between the asymptotic symmetry-resolved density of states (ASRDoS) [5] and SREE was alluded to in [6] for the case of group-like symmetries and extended to the case of categorical symmetries relevant to the untwisted sector in [4]. Therefore, another major aim of this work is to extend this relation for categorical symmetries relevant to the twisted and anyonic sectors.

Central to our developments is to have suitable representations of generalized charged sectors, for these categorical symmetries. Remarkably, it has been realized that the symmetry features of a $(1+1)$-d QFT with a noninvertible symmetry defined by a modular fusion category (MFC) $\mathcal{C}$ are encoded by a $(2+1)$-d topological QFT coupled to it, recently referred to as symmetry topological field theory (SymTFT) [7,8]. This construction is especially useful in understanding what a "generalized charged sector" means in the case of categorical noninvertible symmetries, allowing to properly define partition functions in such charged sectors, which is one of the main results of this work.

Indeed, representations of generalized global symmetries have been recently studied from the perspective of SymTFT in [9]. While this observation on a physical context for $(1+1)$-d CFTs was first noted in [5], authors in [9] found that a similar structure is also valid in the case of $(2+1)$-d theories with fusion 2-category symmetries (see also [10]).

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Our results shed light on the relationship between, categorical symmetries, boundary conditions and entanglement, indicating that an extension of the program to higher dimensions, could provide powerful tools to study topological order and phases.

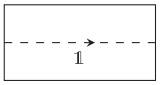
II. RATIONAL CFT BASICS

An RCFT is a 2D CFT with a finite number of primary fields a which are the highest weight integral irreps with respect to its symmetry/chiral algebra $\mathcal{A}$ [11]. These irreps arrange themselves as *Verma modules* $\mathcal{V}_a$ where their levels contain primaries and their descendants. With this, the bulk Hilbert space decomposes as [5,12,13]

$$\mathcal{H}_{\mathbb{1}} = \bigoplus_{a\in\mathcal{C}} \mathcal{V}_a \otimes \overline{\mathcal{V}_a}. \tag{1}$$

The torus partition function of the theory is given as

$$\mathcal{Z}(\mathfrak{q},\bar{\mathfrak{q}}) = \mathrm{tr}_{\mathcal{H}_{\mathbb{1}}}\left[\mathfrak{q}^{L_0-\frac{c}{24}}\bar{\mathfrak{q}}^{\bar{L}_0-\frac{\bar{c}}{24}}\right] = \quad = \sum_a \chi_a(\mathfrak{q})\overline{\chi_a(\mathfrak{q})}, \tag{2}$$



where $\mathbb{1}$ in the subscript of $\mathcal{H}$ denotes that the bulk Hilbert space is the untwisted Hilbert space or trivially twisted by the identity *topological defect line* (TDL), $\mathbb{1}$ (more on this below). $\mathcal{H}_{\mathbb{1}}$ is defined on a time slice along spatial S^1, ($\bar{L}_0$) L_0 are the (anti)holomorphic Virasoro zero modes and ($\bar{c}$) c are the (anti)holomorphic central charges. $\mathfrak{q} = e^{2\pi i\tau}$ where τ is the modular parameter on the torus and $\chi_a(\mathfrak{q})$ denotes the linearly independent characters [14]. The number of characters is less than or equal to the number of primaries 12,15]]. The black box in (2) represents the torus partition function with the temporal and spatial S^1 along the vertical and horizontal axis, respectively.

Other important ingredients of an RCFT are its *modular data*, given by the $S\colon\ \tau \to -\frac{1}{\tau}$ and $T\colon\ \tau \to \tau+1$ matrices [16] and the operator product expansion coefficients which dictate how primaries behave when they are brought close to each other inside correlation functions.

Roughly, the above data also define what is called a MFC: $\mathcal{C}$. Its elements (or simple objects): $a, b, c, d, \ldots \in \mathcal{C}$ are in one-to-one correspondence with the primaries. Furthermore, their *fusion rule/algebra* $a \times d = \sum_c N_{ad}^c c$, gives rise to the operator product expansion coefficients. From these, one can obtain the S and T matrices [17]. This is how an MFC encodes the modular data of an RCFT. The fusion coefficients—in the fusion rule—$N_{ad}^c \in \mathbb{Z}_{\geq 0}$ are given by the Verlinde formula: $N_{ad}^c = \sum_b \frac{S_{ba}S_{bd}S_{bc}^*}{S_{1b}}$, where S_{ab} denotes an element of the S-matrix of the RCFT, with the subscript "1" labeling the vacuum of the theory: $|1\rangle$.

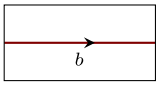
The action of TDLs generate 0-form global symmetries in 2D RCFTs [13]. In a diagonal RCFT, its TDLs b, are in one-to-one correspondence with its bulk primaries, hence the same label for both. Furthermore, in this scenario these TDLs are referred to as Verlinde lines and they commute with all the generators of the chiral algebra, satisfying the same fusion algebra as that of the bulk primaries. They act on the bulk primaries as [5,13,18],

$$b|\phi_a\rangle = \frac{S_{ba}}{S_{1a}}|\phi_a\rangle. \tag{3}$$

III. TORUS PARTITION FUNCTION AND TDL INSERTIONS

Consider inserting a TDL b, along the spatial S^1. This amounts to it acting on the bulk primaries as in (3). Then, we have from (2) [5,13]

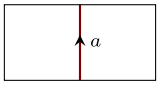
$$\mathcal{Z}^b(\mathfrak{q},\bar{\mathfrak{q}}) = \mathrm{tr}_{\mathcal{H}_{\mathbb{1}}}\left[b\,\mathfrak{q}^{L_0-\frac{c}{24}}\bar{\mathfrak{q}}^{\bar{L}_0-\frac{\bar{c}}{24}}\right] = \quad = \sum_a \frac{S_{ba}}{S_{1a}}\chi_a(\mathfrak{q})\overline{\chi_a(\mathfrak{q})}. \tag{4}$$

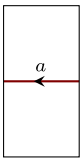


We can also consider inserting a TDL along the temporal S^1. In this case, the Hilbert space will be modified to give rise to a *defect* Hilbert space twisted by the TDL. Then, the twisted torus partition function is given as [18]

$$\mathcal{Z}_a(\mathfrak{q},\bar{\mathfrak{q}}) = \mathrm{tr}_{\mathcal{H}_a}\left[\mathfrak{q}^{L_0-\frac{c}{24}}\bar{\mathfrak{q}}^{\bar{L}_0-\frac{\bar{c}}{24}}\right] = \quad = \quad = \sum_b \frac{S_{\bar{a}b}}{S_{1b}}\chi_b(\tilde{\mathfrak{q}})\overline{\chi_b(\tilde{\mathfrak{q}})}, \tag{5}$$

$$= \sum_{b,c,d} \frac{S_{\bar{a}b}}{S_{1b}} S_{bd}S_{bc}^*\chi_d(\mathfrak{q})\overline{\chi_c(\mathfrak{q})} = \sum_{c,d} N_{ad}^c\chi_d(\mathfrak{q})\overline{\chi_c(\mathfrak{q})},$$





where in the second line above, we have used a modular S transformation to get the following relation: $\mathcal{Z}_a(\mathfrak{q},\bar{\mathfrak{q}}) = \mathcal{Z}^a(\tilde{\mathfrak{q}},\bar{\tilde{\mathfrak{q}}})$ with $\tilde{\mathfrak{q}} = e^{-\frac{2\pi i}{\tau}}$ where $S\colon\ \mathfrak{q} \to \tilde{\mathfrak{q}}$. In the last equality above we have used Eq. (17.7) of [19] along with the assumption that the given RCFT has only self-conjugate irreps. This implies $\bar{a} = a$, where $\bar{a}$ is the orientation reversal line of a. Hence, the S-matrix is real. So, from (5) we get,

$$\mathcal{H}_a = \bigoplus_{c,d} N_{ad}^c \mathcal{V}_c \otimes \overline{\mathcal{V}_d}. \tag{6}$$

IV. SREE FOR INVERTIBLE SYMMETRIES

Before proceeding, let us introduce a pictorial notation. As explained above, a black box would imply a torus partition function (4). Spatial TDL insertion will be denoted by a horizontal maroon arrow and temporal TDL insertion by a vertical maroon arrow, (5) and (7). However, a brown box with blue colored right and left edges will denote the annulus partition function (see figure in the Supplemental Material [20]) rather than the torus partition function. The annulus partition function would include the interval A and at its end points the conformal boundary conditions, labeled by the blue edges (15). Furthermore, we denote the corresponding modular parameter on the annulus as $q \equiv e^{2\pi i\tau}$.

In an RCFT with a discrete invertible symmetry described by an Abelian group G, the bulk Hilbert space $\mathcal{H}_{\mathbb{1}}$ decomposes into various charged sectors $\mathcal{H}_r$ (labeled by irrep r) with the projectors, $P_r\colon \mathcal{H}_{\mathbb{1}} \to \mathcal{H}_r$ given by [6],

$$P_r = \frac{d_r}{|G|}\sum_{g\in G}\chi_r^*(g)\ \xrightarrow{\ g\ }\ ,\forall\ g\in G \tag{7}$$

where d_r is the dimension of the irrep, $|G| \equiv \sum_r d_r^2$ is the volume of the group and $\chi_r^*(g)$ is the character of the element of $g \in G$ in the irrep r. The spatial TDL g acts on states of the bulk Hilbert space and is inserted into the torus partition function as in (4). Then, we insert the spatial region A of length ℓ along a fixed temporal slice and impose G-invariant conformal boundary conditions at the entangling points (see figure in Supplemental Material [20]). This allows for the *topological endability* of g on the conformal boundary (see [21]). Next we conformally map the geometry to that of the annulus (see [20]), to get to the annulus partition function.

With the above construction in mind and with the projectors as in (7), the projected annulus partition function associated with the r-charged sector reads as (see [4,20])

$$\mathcal{Z}_{\alpha\beta}[q^n, r] = \mathrm{tr}[P_r\rho_A^n] = \frac{d_r}{|G|}\sum_{g\in G}\chi_r^*(g)\frac{Z_{\alpha\beta}[q^n, g]}{(Z_{\alpha\beta}[q])^n},$$
$$Z_{\alpha\beta}[q^n, g] = \mathrm{tr}[gq^{n(L_0-c/24)}], \tag{8}$$

where ρ_A denotes the reduced density matrix for the region A. In the above definitions all the traces (over *open string* states) are subjected to explicit boundary conditions α and β on the entangling points of the interval A and the action of TDL g is encoded in the *charged moments* $Z_{\alpha\beta}[q^n, g]$, defined above in the *open string* channel (see figure in Supplemental Material [20]).

Using the boundary CFT approach to EE (see [20]), one may express $Z_{\alpha\beta}[q^n, g]$ in the S-dual channel (or *closed string* channel) in terms of boundary states $|\widetilde{\alpha}\rangle_g$ and $|\widetilde{\beta}\rangle_g$ as $Z_{\alpha\beta}[q^n, g] =_g \langle\widetilde{\alpha}|\tilde{q}^{\frac{1}{n}(L_0-c/24)}|\widetilde{\beta}\rangle_g$, where $\tilde{q} \equiv S(q)$ is the S-transform of q and the subindex g refers to the fact that states $|\widetilde{\alpha}\rangle_g$, $|\widetilde{\beta}\rangle_g$ are conformal Cardy states belonging to the *twisted* Hilbert space $\mathcal{H}_g$ [6,22]. Note that as $Z_{\alpha\beta}[q^n, g]$ is defined through the insertion of g in the annulus partition function (or *open string* channel), we require g to end *topologically* on the boundary of A since it is a symmetry defect [6,21]. This imposes a constraint on the allowed boundary states in the dual S-channel (or, *closed string* channel) [23]. For group-like symmetries the topological endability is equivalent to having G-invariant boundary states: $g'|\widetilde{\alpha}\rangle_e = |\widetilde{\alpha}_e\rangle,\ \ \forall\ \ g' \in G$, where e denotes the identity element of G.

With this, it has been shown [6,24,25] that the SREE defined by [26],

$$\mathcal{S}_n[q, r] = \frac{1}{1-n}\log\frac{\mathcal{Z}_{\alpha\beta}[q^n, r]}{(\mathcal{Z}_{\alpha\beta}[q, r])^n}, \quad \mathcal{S}[q, r] = \lim_{n\to 1}\mathcal{S}_n[q, r], \tag{9}$$

in the $\tilde{q} \to 0 (\ell \gg \epsilon)$ limit reads as

$$\mathcal{S}[q, r] = \frac{c}{3}\log\frac{\ell}{\epsilon} + (g_\alpha^* + g_\beta) + \log\frac{d_r^2}{|G|} + \mathcal{O}\left(\frac{\epsilon}{\ell}\right), \tag{10}$$

where $g_\alpha \equiv \log\langle 1|\widetilde{\alpha}\rangle$ is the *Affleck-Ludwig* boundary entropy. Observing (10), we see that at leading order in the UV cutoff, the entanglement entropy of region A associated with the r-charged sector $\mathcal{S}[q, r]$ is equal for all sectors, that is, independent of r. This is known as *entanglement equipartition* [6,27,28]. For an Abelian symmetry group G, the equipartition holds even at the subleading order, $\mathcal{O}(1)$ while for a non-Abelian symmetry group it breaks at this order (see [6,26,29–31]). Note that, for group-like symmetries generated by the action of Verlinde lines, we would always have an Abelian group since the commutativity of fusion rules implies that the symmetry group is Abelian.

V. CATEGORICAL SREE

In [4], the above group-like SREE was extended to the case of noninvertible symmetries.

While considering the action of Verlinde lines on primaries, one needs to distinguish their action on bulk primaries and on defect primaries. TDLs map bulk primaries to bulk primaries, that is, $b\colon \mathcal{H}_{\mathbb{1}} \to \mathcal{H}_{\mathbb{1}}$ owing to (3). However, their action on defect primaries is given by the map $b\colon \mathcal{H}_a \to \mathcal{H}_c$, that is, twisted sectors are mapped into each other. In [4], categorical symmetry resolution was performed with respect to the Verlinde lines acting on the untwisted sector $\mathcal{H}_{\mathbb{1}}$. In this spirit, we term this symmetry resolution as *untwisted* categorical SREE (CaT-SREE). The two main ingredients needed to do this are the relevant projectors, as in (7) and the notion of topological endability of noninvertible lines. The latter concept was considered in [21] and was properly formalized in [23] from which we only need the notion of $\mathcal{C}$-weakly symmetric boundary

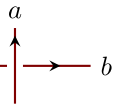
states to define CaT-SREE as was advocated in [4]. Given a category $\mathcal{C}$, a conformal boundary state is $\mathcal{C}$-weakly symmetric if $a\widetilde{|\beta\rangle} = \widetilde{|\beta\rangle} + \ldots, \ \forall \ a \in \mathcal{C}$. If weakly symmetric states exist, we can symmetry resolve with respect to the given category (more details in [20]).

Additionally, we make an important observation: at least one noninvertible line needs to exist in an MFC in order to perform symmetry resolution of entanglement with respect to the action of Verlinde lines—either for the group-like or the categorical case (see the penultimate section in [20]).

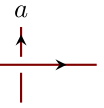
VI. TWISTED CAT-SREE

In this section we generalize [4] by allowing for the action of TDLs on defect primaries. Let us study the spectrum of the defect spaces. It can be computed using (5) which results in the decomposition as given in (6) (see [20]). From this, one needs to construct relevant projectors which map from $\mathcal{H}_{a\neq\mathbb{1}}$ to the subspace $\mathcal{V}_c \otimes \overline{\mathcal{V}_d}$, $\forall \ a, c, d \in \mathcal{C}$. Following [5], we first define, the following two sets of orthogonal projectors:

$$P_a^{(c,*)} \equiv \sum_b S_{1c} S_{bc}^* \quad b \ , \tag{11}$$



$$P_a^{(*,d)} \equiv \sum_b S_{1d} S_{bd}^* \quad b \ . \tag{12}$$

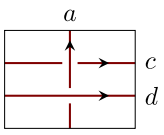


It can be shown using the definitions above (11) and (12), that the sets $\{P_a{}^{(c,*)}\}_{a\in\mathcal{C}}$ and $\{P_a{}^{(*,d)}\}_{a\in\mathcal{C}}$ are individually *complete* and *idempotent* too [5].

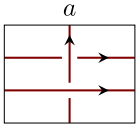
Now, from these two sets, let us define the relevant projectors as $P_a^{(c,d)} \equiv P_a^{(c,*)} P_a^{(*,d)}$ whose action is given as $P_a^{(c,d)}: \ \mathcal{H}_a \to N_{ad}^c \mathcal{V}_c \otimes \overline{\mathcal{V}_d}$ [5]. Note that, when $a = \mathbb{1}$ then $c = d$ due to the decomposition given in (1) for diagonal RCFTs.

From the above, first let us define the corresponding projected torus partition functions as [5],

$$\mathrm{tr}\left[P_a^{(c,d)}\rho\right] \equiv \mathrm{tr}_{\mathcal{H}_a^{(c,d)}}\left[\mathfrak{q}^{L_0-c/24}\,\bar{\mathfrak{q}}^{\bar{L}_0-\frac{c}{24}}\right] =$$



$$= \sum_{b,b'} S_{1c} S_{bc}^* S_{1d} S_{b'd}^*$$

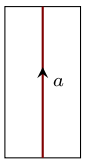


. (13)

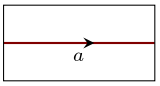
where ρ denotes the full density matrix of the theory [32].

Now b and b' above can be fused and this calculation is present in [20] (see also [5]). Then, we have (with b'' denoting the fused line from fusion of b and b'),

$$\mathrm{tr}\left[P_a^{(c,d)}\rho\right] \overset{b''\to\mathbb{1}}{\sim} \frac{N_{ad}^c}{|\mathcal{C}|}\frac{d_c\,d_d}{d_a}$$



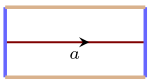
$$\overset{\text{under } S^{-1}}{=} \frac{N_{ad}^c}{|\mathcal{C}|}\frac{d_c\,d_d}{d_a}$$



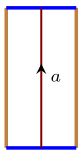
(14)

where we have used $|\mathcal{C}| = \frac{1}{S_{11}^2}$ (see [20] for more details). Note that, here, first we are constructing the torus partition function as above and then next from it the annulus partition function will be constructed below by taking the interval A of length ℓ in (14) (see [20] for details),

$$\mathcal{Z}^a_{\alpha\beta}[q,(c,d)] \equiv \mathcal{Z}^a[q,(c,d)] \sim \frac{N_{ad}^c}{|\mathcal{C}|}\frac{d_c\,d_d}{d_a}$$



$$\overset{\text{under } S}{=} \frac{N_{ad}^c}{|\mathcal{C}|}\frac{d_c\,d_d}{d_a}$$



$$= \frac{N_{ad}^c}{|\mathcal{C}|}\frac{d_c\,d_d}{d_a}\left({}_a\langle\widetilde{\beta_1}|\tilde{q}^{(L_0-\frac{c}{24})}|\widetilde{\beta_2}\rangle_a\right)$$

$$\overset{\tilde{q}\to 0}{\sim} \frac{N_{ad}^c}{|\mathcal{C}|}\frac{d_c\,d_d}{d_a} g^a_{\beta_1}\bar{g}^a_{\beta_2}\tilde{q}^{h_0^a-\frac{c}{24}},$$

(15)

where in the first line we have explicitly introduced the weakly symmetric boundary conditions α, β to denote that we are computing the annulus partition functions with these boundary conditions. For simplicity, we drop the notation of α, β from here on [38]. Also, $g^a_{\beta_i} \equiv_a \widetilde{\langle\beta_i|1\rangle}_a$ can be referred to as the *a-twisted boundary entropy*. It computes the overlap of the twisted Cardy state $\widetilde{|\beta_i\rangle}$ with the twisted vacuum in $\mathcal{H}_a$—denoted by $|1\rangle_a$—which is the defect scalar primary in $\mathcal{H}_a$ with the lowest conformal dimension h_0^a (see [20] for an untwisted analogue of the boundary entropy expression).

We employ a simplification from here on, that is to choose the same and fixed boundary conditions—across all sectors—in (15), at both the entangling points (see figure in Supplemental Material [20]), implying, $\widetilde{|\beta_1\rangle} = \widetilde{|\beta_2\rangle}$.

Now using (9) and (15), we can compute $(\mathcal{Z}^a[q,(c,d)])^n$ and $\mathcal{Z}^a[q^n,(c,d)]$ for the replication $q \to q^n$. From this, using (9), we get for the twisted sector SREE

$$\mathcal{S}^a[q,(c,d)] = \frac{c-24h_0^a}{3}\log\frac{\ell}{\epsilon} + \log\left[\frac{N_{ad}^c d_c d_d}{d_a|\mathcal{C}|}(g^a_\beta)^2\right]. \tag{16}$$

(16) is kind of a correspondence between ASRDoS as in Eqs. (3.42) or (4.24) of [5] and twisted CaT-SREE. The appearance of h_0^a in (16) is reminiscent of the fact that we are dealing with defect Hilbert spaces where the corresponding irreps are labeled by the eigenvalues of the T-matrix which happen to be the conformal dimensions [5,16]. We will comment more on this in [39]. Note that, (16) with $a = \mathbb{1}$ implies $c = d$. This now reproduces the main result of [4] that is Eq. (33).

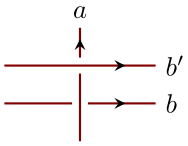
VII. GENERALIZED ANYONIC SECTORS AND SYMTFT

For $(1+1)$-d CFTs with noninvertible symmetries, the notion of charged sectors arises by inserting TDLs in the time direction and then decomposing the resulting Hilbert space into irreducible representations, each of which corresponds to a generalized charged sector. As mentioned above, TDLs map local fields to local fields [13] and their action on a twisted defect field is described by the so-called *lasso action* [5,13] under which the resulting defect field might belong to a different defect Hilbert space [5]. When this is the case, one needs to consider the *total* Hilbert space as, $\mathcal{H}_{\text{total}} = \bigoplus_a \mathcal{H}_a, a \in \mathcal{C}$ with $a\colon \mathcal{H}_{a\neq\mathbb{1}} \to \mathcal{H}_b$ in general. Then $\mathcal{H}_{\text{total}}$ can be decomposed into various irreps μ under the action of TDLs given by the so-called *tube algebra* of $\mathcal{C}$ [denoted Tube($\mathcal{C}$)]. These irreps are referred to as *μ-anyonic sectors* and, remarkably, are in one-to-one correspondence with the bulk anyons μ of the Drinfeld center $\mathcal{D}(\mathcal{C})$ (quantum double) of the $(2+1)$-d SymTFT [5]. Concretely, in the case of two-dimensional RCFTs, $\mathcal{D}(\mathcal{C})$ is given by $\mathcal{C}\boxtimes\bar{\mathcal{C}}$ and thus, a bulk anyon $\mu \in \mathcal{D}(\mathcal{C})$ is given by $\mu = c \otimes \bar{d}$ with c, d denoting the simple objects in $\mathcal{C}$. For RCFTs with self-conjugate irreps, we further have, $\mu = c \otimes d$.

VIII. ANYONIC CAT-SREE

The above-defined μ labels the irreps of $\mathcal{C}$ and are in one-to-one correspondence with the generalized charges of [9] (see also [10]). Following [5] we define the relevant projectors as,

$$P^{\mu} \equiv P^{(c,d)} = \sum_a P_a^{(c,*)} P_a^{(*,d)} = \sum_{a,b,b'} S_{0c} S^*_{bc} S_{0d} S^*_{b'd}$$



(17)

This amounts to saying that the charged moments we are looking for are given by

$$\mathcal{Z}[q,\mu] \equiv \mathcal{Z}[q,(c,d)] \equiv \sum_{a\in\mathcal{C}} \mathcal{Z}^a[q,(c,d)]. \tag{18}$$

In terms of the anyonic sectors we can rewrite (15) as

$$\mathcal{Z}^a[q,\mu] = \langle \mu, a\rangle \frac{d_\mu}{|\mathcal{C}|} \frac{(g_\beta^a)^2}{d_a} \tilde{q}^{h_0^a - c/24} \tag{19}$$

where d_μ denotes the dimension of the bulk anyon and we have used the relation [see Eq. (4.47) of [5]] $d_\mu \langle \mu, a\rangle = N_{ca}^d d_c d_d$. Now $\langle \mu, a\rangle$ is the projection of the bulk anyon μ onto the boundary line a [5]. Furthermore, to get (19) from (15) we also require $N_{ad}^c = N_{ca}^d$ which is true for self-conjugate irreps (see the last section in [20]).

Using the above we can write (18) as,

$$\begin{aligned}\mathcal{Z}[q,\mu] = \mathcal{Z}[q,(c,d)] &= \frac{1}{|\mathcal{C}|}\sum_{a\in\mathcal{C}} \frac{d_\mu \langle \mu, a\rangle}{d_a} (g_\beta^a)^2 \tilde{q}^{h_0^a - c/24} \\ &= \frac{1}{|\mathcal{C}|}\Bigg[\underbrace{d_\mu \langle \mu, \mathbb{1}\rangle g_\beta^2 \tilde{q}^{(-\frac{c}{24})}}_{\text{leading in } \tilde{q}\to 0} + \underbrace{\sum_{a\neq\mathbb{1}} \frac{(g_\beta^a)^2 d_\mu \langle \mu, a\rangle}{d_a} \tilde{q}^{h_0^a - \frac{c}{24}}}_{\text{sub-leading in } \tilde{q}\to 0}\Bigg] \\ &\overset{\tilde{q}\to 0}{\sim} \frac{d_\mu}{|\mathcal{C}|} \langle \mu, \mathbb{1}\rangle g_\beta^2 \tilde{q}^{(-\frac{c}{24})} = \frac{d_c^2}{|\mathcal{C}|} g_\beta^2 \tilde{q}^{(-\frac{c}{24})},\end{aligned} \tag{20}$$

where the last equality is obtained noting that we have $\langle \mu, a\rangle = N_{ca}^d$ (see [5]). This implies $\langle \mu, \mathbb{1}\rangle = N_{\mathbb{1}c}^c = 1$. As $c \times \mathbb{1} = \sum_d N_{c\mathbb{1}}^d d = \sum_d \delta_{c,d} d = c$, implying, $\langle \mu, \mathbb{1}\rangle = 1$.

In (20), $\mu = c \otimes d$ for any fixed $c \in \mathcal{C}$ with all $d \in \mathcal{C}$. This equation holds only for diagonal RCFTs since for non-diagonal theories, the bulk Hilbert space $\mathcal{H}_{\mathbb{1}}$ would have the following decomposition in general: $\mathcal{H}_{\mathbb{1}} = \bigoplus_{c,d} M_{c,d} \mathcal{V}_c \otimes \bar{\mathcal{V}}_d$. One would then need to construct the untwisted projectors $\{P_{\mathbb{1}}^{(c,d)}\}_{c,d\in\mathcal{C}}$ as *nondiagonal* generalizations of the untwisted projectors $\{P_{\mathbb{1}}^{(c,c)}\}_{c\in\mathcal{C}}$ as constructed in Eq. (3.8) of [5].

IX. BREAKING OF ENTANGLEMENT EQUIPARTITION

Now we state the important observation that, with respect to the anyonic sectors, *entanglement equipartition breaks* at the subleading order in the UV cutoff ϵ in the case of categories with noninvertible TDLs as opposed to categories with only invertible TDLs.

We can consider the replication $q \to q^n$ and the n^{th} power of $\mathcal{Z}[q,\mu] = \mathcal{Z}[q,(c,d)]$ as in (20), from which, we obtain the following SREE for the anyonic sectors with $\mu = c \otimes d$ [using (9) and noting that $\tilde{q} = e^{-2W} \sim e^{-4\log\frac{\ell}{\epsilon}}$]:

$$\mathcal{S}[q,(c,d)] = \frac{c}{3}\log\frac{\ell}{\epsilon} + \log\frac{d_c^2}{|\mathcal{C}|} + \log g_\beta^2 + \mathcal{O}\left(\frac{\epsilon}{\ell}\right). \tag{21}$$

(21) is kind of a correspondence between ASRDoS as in Eq. (4.23) of [5] and anyonic CaT-SREE; we will comment

more on this in [39]. Now let us compare (21) to Eq. (33) of [4]. In the untwisted case, "charged sectors" (c,c) correspond to the Verma modules $\mathcal{V}_c \otimes \overline{\mathcal{V}_c}$ while in the anyonic case, the "charged sectors" are the irreps of the tube algebra, that is, $\mu \in \mathrm{Rep}(\mathrm{Tube}(\mathcal{C})) \equiv \mathcal{D}(\mathcal{C})$. So, the cardinalities of the sectors differ in both cases and the anyonic CaT-SREE correctly accounts for the total EE associated with the region A by summing over the μ sectors [which differs from a sum over untwisted sectors (c,c)] as

$$\mathcal{S}[q] = \sum_{\mu \in \mathcal{D}(\mathcal{C})} p[\mu]\mathcal{S}[q,\mu] - \sum_{\mu \in \mathcal{D}(\mathcal{C})} p[\mu]\log p[\mu], \quad (22)$$

where $p[\mu] = \mathrm{tr}[P^\mu \rho_A]$ is the probability of measuring the representation μ within block A. The first term, known as configurational EE, depends on the entropy of each μ-charged sector. The second is due to the fluctuations of the charge within A [6] and both are experimentally accessible [40].

Finally, we note that if $\mathcal{C}$ contains only invertible TDLs [41] then $\mathcal{D}(\mathcal{C})$ will contain bulk anyons μ with $d_\mu = 1, \ \forall \ \mu \in \mathcal{C}\boxtimes\bar{\mathcal{C}}$ (by construction). Then, it follows that entanglement equipartition holds even at subleading order $\mathcal{O}(1)$ from (21).

However, if $\mathcal{C}$ contains at least one noninvertible line, then at least one generalized irrep μ has $d_\mu > 1$ and we observe a *breaking of entanglement equipartition* at subleading order.

X. TRICRITICAL ISING MODEL

To illustrate the above results we consider the tricritical Ising model which is a diagonal RCFT with central charge $c = \frac{7}{10}$ and six Virasoro primaries and six corresponding TDLs: $\mathcal{C} = \{\mathbb{1}, \eta, W, N, \eta W, WN\}$ [12,13].

First, we note that $\mathcal{C}_2 \equiv \{\mathbb{1}, \eta\}$ is an invertible symmetry subcategory of $\mathcal{C}$—with $\mathbb{Z}_2$ fusion rule—which admits the weakly symmetric boundary states $|N\rangle$ and $|WN\rangle$. This can be seen using Eq. (2.11) of [23] and the nontrivial fusion rules $\eta|N\rangle = |N\rangle$ and $\eta|WN\rangle = |WN\rangle$. So, we can consider $\mathbb{Z}_2$ symmetry resolution with respect to the anyonic sectors of the subcategory $\mathcal{C}_2$, which are $\mu \in \mathcal{D}(\mathbb{Z}_2) = \{\mathbb{1}\otimes\mathbb{1}, \mathbb{1}\otimes\eta, \eta\otimes\mathbb{1}, \eta\otimes\eta\}$. Since $d_\mathbb{1} = d_\eta = 1$, applying (21) to these sectors we obtain,

$$\begin{aligned}\mathcal{S}[q,(\mathbb{1},\mathbb{1})] &= \mathcal{S}[q,(\mathbb{1},\eta)] = \mathcal{S}[q,(\eta,\mathbb{1})] = \mathcal{S}[q,(\eta,\eta)] \\ &= \frac{7}{30}\log\frac{\ell}{\epsilon} - \log(2) + \log g_\beta^2 + \mathcal{O}\left(\frac{\epsilon}{\ell}\right), \quad (23)\end{aligned}$$

where g_β depends upon the choices of weakly symmetric boundary conditions, either $|N\rangle$ or $|WN\rangle$ at both the entangling points. So, we see an equipartition of entanglement—up to order $\mathcal{O}(1)$—across all anyonic sectors for the invertible subcategory $\mathcal{C}_2$.

Next we note that $\mathcal{C}_{\mathrm{Fib}} \equiv \{\mathbb{1}, W\}$ is a noninvertible MFC of $\mathcal{C}$—with Fibonacci fusion rule—which admits the weakly symmetric boundary states $|W\rangle$, $|\eta W\rangle$ and $|WN\rangle$. This can be seen using Eq. (2.11) of [23] and the nontrivial fusion rules $W|W\rangle = |W\rangle + |\mathbb{1}\rangle$, $W|\eta W\rangle = |\eta W\rangle + |\eta\rangle$ and $W|WN\rangle = |WN\rangle + |N\rangle$. So, we can consider Fibonacci symmetry resolution with respect to the anyonic sectors of the subcategory $\mathcal{C}_{\mathrm{Fib}}$, which are $\mu \in \mathcal{D}(\mathcal{C}_{\mathrm{Fib}}) = \{\mathbb{1}\otimes\mathbb{1}, \mathbb{1}\otimes W, W\otimes\mathbb{1}, W\otimes W\}$. Since $d_W = \zeta \equiv \frac{1+\sqrt{5}}{2}$, applying (21) to these sectors we obtain,

$$\begin{aligned}\mathcal{S}[q,(\mathbb{1},\mathbb{1})] &= \frac{7}{30}\log\frac{\ell}{\epsilon} - \log(1+\zeta^2) + \log g_\beta^2 + \mathcal{O}\left(\frac{\epsilon}{\ell}\right) \\ &= \mathcal{S}[q,(\mathbb{1},W)], \quad (24)\end{aligned}$$

$$\begin{aligned}\mathcal{S}[q,(W,W)] &= \frac{7}{30}\log\frac{\ell}{\epsilon} + \log\frac{\zeta^2}{1+\zeta^2} + \log g_\beta^2 + \mathcal{O}\left(\frac{\epsilon}{\ell}\right) \\ &= \mathcal{S}[q,(W,\mathbb{1})], \quad (25)\end{aligned}$$

where, as before, g_β depends upon the chosen choices of weakly symmetric boundary conditions, either $|W\rangle$, $|\eta W\rangle$ or $|WN\rangle$ at both the entangling points. So, as discussed above, we see that for the noninvertible subcategory $\mathcal{C}_{\mathrm{Fib}}$, the equipartition of entanglement breaks at $\mathcal{O}(1)$.

XI. DISCUSSION AND OUTLOOK

We have provided a full symmetry resolution for the EE of an interval in a $(1+1)$-d RCFT with a noninvertible symmetry defined through an MFC. For this, we have used suitable notions of generalized charged sectors of the MFC, which are given by the anyons of the associated $(2+1)$-d SymTFT, to build the projected partition functions in such generalized charged sectors that define the anyonic CaT-SREE.

Our results apply to diagonal RCFT with a unique vacuum in the bulk Hilbert space with all of its irreps being self-conjugate. These conditions can be relaxed to allow for the application of the anyonic CaT-SREE to more general scenarios such as nondiagonal RCFTs [42], arbitrary fusion categories [5] or nonunitary theories, to be considered in an upcoming work [39].

Note added. During the final stages of preparation of this manuscript, we were made aware of partly related developments in [33–37].

ACKNOWLEDGMENTS

We express our sincerest gratitude to Christian Northe, Daniel Grumiller, Germán Sierra, Ho Tat Lam, Iñaki García-Extebarria, Jishu Das, Mathew Bullimore, Michele Fossati, Nabil Iqbal, Naveen Balaji Umasankar, Rajath Radhakrishnan, Saranesh Prembabu, Shovon Biswas, Statis Vitouladitis, Thomas Bartsch, Timmavajjula Venkata Karthik, and Tin Sulejmanpasic for many insightful discussions. We thank Brandon Rayhaun, Yichul Choi, Yunqin

Zheng, Christian Copetti, Daniel Pajer, Lakshya Bhardwaj, Sakura Schafer-Nameki, Jared Heymann, Thomas Quella, Angel Uranga, Jesús Huertas and Iñaki García Etxebarria for agreeing to coordinate the release of partially related upcoming works with us. The work of A. D. is supported by the STFC Consolidated Grant No. ST/T000600/1—"Particle Theory at the Higgs Centre." J. M.-V. thank the financial support of Spanish Ministerio de Ciencia e Innovación PID2021-125700NAC22. The work of P. S.-B. is supported by Fundación Séneca, Agencia de Ciencia y Tecnología de la Región de Murcia, Grant No. 21609/FPI/21.

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Chapter 5

Entropic order parameters for categorical symmetries in 2D-CFT

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Article

Entropic Order Parameters for Categorical Symmetries in 2D-CFT

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Abstract: In this work, we propose an information theoretic order parameter able to characterize the presence and breaking of categorical symmetries in $(1+1)$-d rational conformal field theories (RCFTs). Specifically, we compute the quantum relative entropy between the ground states of RCFTs representing the critical point of phase transitions between different symmetry-broken phases of theories with categorical symmetries, and their symmetrized versions. We find that, at leading order in the high temperature limit, this relative entropy only depends on the expectation values of the quantum dimensions of the topological operators implementing the categorical symmetry. This dependence suggests that our proposal can be used to characterize the different broken phases of $(1+1)$-d theories with categorical symmetries.

Keywords: relative entropy; categorical symmetries; conformal field theory; symmetry breaking

1. Introduction

The last few years have witnessed remarkable developments in the notion of a global symmetry in theoretical physics. The major idea comes from realizing that every symmetry operation can be associated with a topological operator [1]. In this new framework, symmetries are not only captured by a discrete or continuous group, but also by more general mathematical structures such as higher groups and fusion categories. This new viewpoint, known as categorical symmetry, has led to a paradigm shift and fostered new directions in the study of quantum field theories (QFT) (see [2] for a comprehensive and pedagogical review on these developments).

Global symmetries are central to the understanding of physics at low energies. As a fact, they characterize the different phases of a theory in terms of spontaneously symmetry-broken (SSB) phases and the corresponding phase transitions between such phases. For group-like symmetries, this refers to what is known as the Landau paradigm. Due to the extension of the concept of global symmetry represented by categorical symmetries, there has been a recent interest in the study of phases in QFT constrained by these categorical symmetries [3,4]. The aim is thus to generalize the standard Landau theory to a categorical Landau paradigm in which novel phases and phase transitions may arise. In this scenario, for instance, it has been shown that for $(1+1)$-d gapped phases with categorical symmetries, contrary to the case of the usual group-like symmetries, the different vacua of a gapped phase may be physically distinguishable [3]. This depends on the properties of interfaces between two distinct vacua which are given by the topological line defects of the non-invertible symmetry.

In order to provide new tools to illuminate the role of generalized categorical symmetries in QFT, here, following [5], we propose an order parameter able to characterize the presence and breaking of these symmetries and, thus, the new phases of matter related to those symmetries. This order parameter can be naturally defined using information theoretic quantities and thus can be cast as an entropic order parameter.

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Citation: Molina-Vilaplana, J.; Saura-Bastida, P.; Sierra, G. Entropic Order Parameters for Categorical Symmetries in 2D-CFT. *Entropy* **2024**, *26*, 1064. https://doi.org/10.3390/e26121064

Academic Editors: Olalla Castro-Alvaredo and David X. Horvath

Received: 23 September 2024
Revised: 25 November 2024
Accepted: 3 December 2024
Published: 6 December 2024

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Specifically, we study a concrete class of $(1+1)$-d QFT known as rational conformal field theories (RCFTs), representing the critical point of phase transitions between the different $(1+1)$-d symmetry-broken phases of theories with categorical symmetries, and the entropic order parameter proposed here allows us to focus on the vacuum states of these theories, which are defined on a flat Euclidean spacetime. Following [5], we use the information theoretic quantity known as the quantum relative entropy, which intuitively, quantifies the difference between two quantum states [6]. It is known that the use of relative entropy in the context of QFT poses several advantages with respect to other entropic quantities, such as the entropy [5,7,8]. Namely, all other information quantities can be derived from it.

The paper is structured as follows: In Section 2, we present a brief overview of the basic material covering RCFTs and topological operators implementing global categorical symmetries. In Section 3, we introduce a generalization of the concept of symmetrization of quantum states for the case of modular fusion categories (MFCs) in RCFTs. Then, following [5], the relative entropy between density matrices and their symmetrizations over an MFC, is proposed as an entropic order parameter. In Section 4, we compute the relative entropy between the ground states of RCFTs representing the critical point of phase transitions between different symmetry-broken phases of theories with categorical symmetries and their symmetrized versions. We find that at leading order, this relative entropy only depends on the expectation values of the TDLs or quantum dimensions of the topological operators implementing the symmetry. Finally, we present our conclusions and perspectives for extensions of this work in Section 5.

2. Background

In this section, we review some very basic facts about RCFTs, and topological operators implementing categorical symmetries in these theories. A more comprehensive treatment of these topics can be found in [9].

We consider diagonal RCFTs with a chiral algebra $\mathcal{A}$ and its modular tensor category (MTC) $\mathcal{C}$ given in terms of simple objects $\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots \in \mathcal{C}$.

In these theories, topological lines are in one-to-one correspondence with its bulk primaries and are called Verlinde lines. Those act on the bulk primaries as [9],

$$\mathsf{b}|\phi_a\rangle = \frac{S_{\mathsf{ba}}}{S_{\mathsf{1a}}}|\phi_a\rangle, \tag{1}$$

where S_{ab} denotes an element of the unitary S-matrix of the RCFTs.

The TDLs $\mathsf{a} \in \mathcal{C}$, satisfy the same fusion algebra as that of the bulk primaries, that is

$$\mathsf{a} \times \mathsf{b} = \sum_{c \in \mathcal{C}} N_{\mathsf{ab}}^{\mathsf{c}}\, \mathsf{c}\,, \tag{2}$$

where $N_{\mathsf{ab}}^{\mathsf{c}} \in \mathbb{Z}_{\geq 0}$ are the fusion coefficients given by the Verlinde formula [10]

$$N_{\mathsf{ab}}^{\mathsf{c}} = \sum_{\mathsf{d} \in \mathcal{C}} \frac{S_{\mathsf{da}} S_{\mathsf{db}} S_{\mathsf{dc}}^*}{S_{\mathsf{1d}}}\,. \tag{3}$$

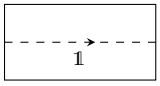
An important property of any TDL $\mathsf{b} \in \mathcal{C}$ is its quantum dimension d_{b}. This is defined as the vacuum expectation value

$$\langle \mathbb{1}|\mathsf{b}|\mathbb{1}\rangle = \frac{S_{\mathsf{b}\mathbb{1}}}{S_{\mathbb{1}\mathbb{1}}} \equiv d_{\mathsf{b}} \tag{4}$$

In any unitary CFT with a unique vacuum, $d_{\mathsf{b}} \geq 0$ [9].

We consider the torus partition function (with the temporal S^1 along the vertical axis with length T and the spatial S^1 along the horizontal axis with length L) given by

$$Z[q,\bar{q}] = \mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) = \tag{5}$$

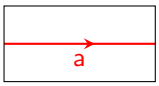


where $q = e^{2\pi i\tau}$, and $\tau = iT/L$, the modular parameter on the torus, so, $q = e^{-2\pi(T/L)}$. The density matrix ρ of the ground state is given thus by

$$\rho = \frac{q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}}{Z[q,\bar{q}]}\,. \tag{6}$$

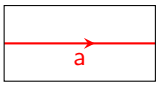
In this work, we will consider insertions of TDLs a along the spatial S^1 which can be depicted as follows:

$$Z[q,\bar{q},\mathsf{a}] = \mathrm{Tr}\left(\mathsf{a}\, q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) = \tag{7}$$

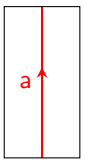


which, from here in advance we simply denote by $Z[q,\mathsf{a}]$. It is important to note that

$$Z[q,\mathsf{a}] = \tag{8}$$



$=$



$= Z^{\mathsf{a}}[\tilde{q}]\,,$

where $\tilde{q}$ is obtained under a modular S-transformation $\tau \to -\frac{1}{\tau}$ and that, in the limit in which we work here, $L \gg T$, hence $\tilde{q} \to 0$,

$$\frac{Z[q,\mathsf{a}]}{Z[q,\mathbb{1}]} \sim \tilde{q}^{\Delta_1^{\mathsf{a}}} \to 0\,, \tag{9}$$

where $\Delta_1^{\mathsf{a}} > 0$ is the scaling dimension of the vacuum corresponding to the vacuum of the a-twisted sector and taking $\Delta_1^{\mathbb{1}} = 0$).

Generically, when a topological operator is defined along the whole space at a fixed time, it amounts to a conserved operator that acts on the Hilbert space $\mathcal{H}$. If the topological operator is inserted in the time direction and localized in one of the spatial directions, the topological operator is a defect that modifies the quantization. The modified quantization gives rise to a twisted Hilbert space [11]. We will not consider the later insertions in this work.

3. Averages over Categories and Relative Entropy as Entropic Order Parameter

In this section we introduce a generalization of averages over groups for the case of modular fusion categories (MFCs) in RCFTs. We first review the case of Abelian groups and then we introduce the generalization. Then, following [5], the relative entropy between density matrices and their averages over an MFC, is proposed as an entropic order parameter.

3.1. Group and Category Averages

In order to fix concepts, let us first recall the symmetrization of ρ as the average over a discrete Abelian group G of the transformed density matrix $g\,\rho\,g^{-1}$, that is,

$$\rho_G = \frac{1}{|G|}\sum_{g\in G} g\,\rho\,g^{-1}, \tag{10}$$

where $|G|$ is the normalized measure of the group, that can be generalized to generic compact Lie groups. Equation (10) implements a map between density matrices known as

conditional expectation [5,6]. In essence, this map extracts the portion of ρ that remains unchanged under the action of the group G. The density matrix ρ_G is by construction symmetric under G and has trace one. Note that ρ is symmetric if and only if $\rho = \rho_G$. Therefore, comparing the two states ρ and ρ_G would lead naturally to probing (spontaneous or explicit) symmetry-breaking [5].

Here, we introduce the average over a category that can be naturally defined as

$$\rho_{\mathcal{C}} = \frac{1}{\mathcal{N}} \sum_{\mathsf{a} \in \mathcal{C}} \mathsf{a}\, \rho\, \bar{\mathsf{a}}\,, \tag{11}$$

where $\bar{\mathsf{a}}$ is the conjugate of a, that is $\mathsf{a} \times \bar{\mathsf{a}} = \mathbb{1} \oplus \ldots$. The constant $\mathcal{N}$ is fixed imposing that $\rho_{\mathcal{C}}$ is a density matrix,

$$\mathrm{Tr}\rho_{\mathcal{C}} = 1 \Longrightarrow \mathcal{N} = \sum_{\mathsf{a},\mathsf{b} \in \mathcal{C}} N_{\mathsf{a}\bar{\mathsf{a}}}^{\mathsf{b}} \mathrm{Tr}[\rho\, \mathsf{b}] \tag{12}$$

For group-like invertible symmetries we will have

$$N_{\mathsf{a}\bar{\mathsf{a}}}^{\mathsf{b}} = \delta_{\mathsf{b}\mathbb{1}} \Longrightarrow \mathcal{N} = \sum_{\mathsf{a} \in \mathcal{C}} \mathrm{Tr}[\rho \mathbb{1}] = |\mathcal{C}| \tag{13}$$

and we will recover the standard definition of ρ_G. However, unlike the case of a group, this map generally does not implement a conditional expectation. Despite this, $\rho_{\mathcal{C}}$ in Equation (11) is a well defined quantum state.

To see this, we refer to [12] where the authors show that non-invertible topological lines generally act on local operators through quantum operations or quantum channels, that is, through completely positive maps between density matrices. In this framework, the operation in Equation (11) should be viewed as a combination of such linear operations with conditional expectations corresponding to specific instances of these general class of quantum operation. In addition, we note that due to the positivity of ρ, the expectation value

$$\langle \psi | \rho_{\mathcal{C}} | \psi \rangle = \frac{1}{\mathcal{N}} \sum_{\mathsf{a} \in \mathcal{C}} \langle \psi\, | \mathsf{a}\, \rho\, \bar{\mathsf{a}}\, | \psi \rangle \geq 0\,, \tag{14}$$

and thus $\rho_{\mathcal{C}}$ is positive. Finally, the condition that a map preserves total probability thus describing the entirety of an operation's outcome rather than a specific subset corresponds to trace-preserving completely positive maps, a property ensured by Equation (13). These conditions collectively ensure that $\rho_{\mathcal{C}}$ can be used later in the context of relative entropies. This being said, it is important to remark that a systematic study clarifying the extent to which Equation (11) represents the invariant part of the density matrix under the non-invertible symmetry is needed. This point is significant and merits further investigation, which we plan to address in a future work, remarking that it is possible to define conditional expectations for categorical symmetries [13], which are not of the form of Equation (11).

3.2. Relative Entropy as an Entropic Order Parameter

The relative entropy between two reduced density matrices ρ, σ is given by

$$S(\rho||\sigma) = \mathrm{Tr}\ \rho \log \rho - \mathrm{Tr}\ \rho \log \sigma\,, \tag{15}$$

that can be derived from

$$S(\rho||\sigma) = \lim_{n \to 1} \frac{1}{n-1} \log\left(\frac{\mathrm{Tr}[\rho^n]}{\mathrm{Tr}[\rho\, \sigma^{n-1}]} \right). \tag{16}$$

Relevant properties of this quantity can be found in [6]. The relative entropy can be operationally interpreted as follows: given σ, the probability p of mistaking ρ for σ after N well-designed experimental measurements decays with N as $p \sim e^{-N S(\rho||\sigma)}$, which endorses the relative entropy as an experimentally accessible quantity.

Here, we propose entropic order parameters that allow us to better understand the symmetry breaking of categorical symmetries in 2D CFTs [3]. Following [5], we investigate the behavior of the relative entropy $S(\rho_G||\rho)$ (resp. $S(\rho_{\mathcal{C}}||\rho)$) between the ground states ρ of the theory and their averaged versions ρ_G (resp. $\rho_{\mathcal{C}}$). It is expected that $S(\rho_G||\rho)$ (resp. $S(\rho_{\mathcal{C}}||\rho)$) explicitly depend on the vacuum expectation values of the TDLs implementing the symmetries under consideration [5]. For the group-like invertible symmetries, it is known that the relative entropy vanishes if the state ρ is invariant under the action of symmetry, i.e., ρ is an unbroken symmetry vacuum, and then the two states that the relative entropy compares are identical. Nevertheless, if the group-like symmetry is broken, the relative entropy between these states must be necessarily non-zero. Hence, the usefulness of this concept as an order parameter in those cases.

Our aim here is to extend this program to the case of symmetries given by modular fusion categories in RCFTs.

4. Relative Entropy in CFTs with Categorical Symmetries

Here we compute the relative entropy between the category-averaged density matrix $\rho_{\mathcal{C}}$ and ρ defined by

$$S(\rho_{\mathcal{C}}||\rho) = \lim_{n\to 1} \frac{1}{n-1} \log\left[\frac{\mathrm{Tr}\,[\rho_{\mathcal{C}}^n]}{\mathrm{Tr}\,[\rho_{\mathcal{C}}\,\rho^{n-1}]}\right]. \tag{17}$$

We focus first on the term $\mathrm{Tr}\left[\rho_{\mathcal{C}}\,\rho^{n-1}\right]$. We note that all $\mathsf{a} \in \mathcal{C}$ are topological and therefore, they are free to move along the Euclidean time direction between the sheets of the replicated manifold under continuous deformations. Indeed, these movements allow to fuse them. With this, one may write the term of interest as

$$\begin{aligned}
\mathrm{Tr}\left[\rho_{\mathcal{C}}\,\rho^{n-1}\right] &= \frac{1}{\mathcal{N}}\sum_{\mathsf{a}\in\mathcal{C}} \mathrm{Tr}\left[\mathsf{a}\rho\bar{\mathsf{a}}\rho^{n-1}\right] = \frac{1}{\mathcal{N}}\sum_{\mathsf{a}\in\mathcal{C}} \mathrm{Tr}\left[\rho^n \mathsf{a}\bar{\mathsf{a}}\right] \\
&= \frac{1}{\mathcal{N}}\sum_{\mathsf{a},\mathsf{b}\in\mathcal{C}} N_{\mathsf{a}\bar{\mathsf{a}}}^{\mathsf{b}}\,\mathrm{Tr}\left[\rho^n \mathsf{b}\right] = \frac{1}{\mathcal{N}}\sum_{\mathsf{a},\mathsf{b},\mathsf{c}\in\mathcal{C}} \frac{(S_{\mathsf{ac}})^2}{S_{1\mathsf{c}}} S_{\mathsf{bc}}\,\mathrm{Tr}\left[\rho^n \mathsf{b}\right] \\
&= \frac{1}{\mathcal{N}}\sum_{\mathsf{b},\mathsf{c}\in\mathcal{C}} \left(\sum_{\mathsf{a}\in\mathcal{C}} (S_{\mathsf{ac}})^2\right) \frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}}\,\mathrm{Tr}\left[\rho^n \mathsf{b}\right] = \frac{1}{\mathcal{N}}\sum_{\mathsf{b},\mathsf{c}\in\mathcal{C}} \frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}}\,\mathrm{Tr}\left[\rho^n \mathsf{b}\right],
\end{aligned} \tag{18}$$

where we have used the Verlinde formula and the fact that $\left(\sum_{\mathsf{a}\in\mathcal{C}} (S_{\mathsf{ab}})^2\right) = 1$.

As we are interested in exploring the asymptotic limit $\tilde{q} \to 0$, it is interesting to write the above result as

$$\begin{aligned}
\mathrm{Tr}\left(\rho_{\mathcal{C}}\rho^{n-1}\right) &= \frac{1}{\mathcal{N}}\left(\sum_{\mathsf{c}\in\mathcal{C}} \mathrm{Tr}\left[\rho^n\right] + \sum_{\mathsf{b}\neq 1,\mathsf{c}\in\mathcal{C}} \frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}}\,\mathrm{Tr}\left[\rho^n \mathsf{b}\right]\right) \\
&= \frac{|\mathcal{C}|}{\mathcal{N}}\mathrm{Tr}\left[\rho^n\right] + \frac{1}{\mathcal{N}}\sum_{\mathsf{b}\neq 1,\mathsf{c}\in\mathcal{C}} \frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}}\,\mathrm{Tr}\left[\rho^n \mathsf{b}\right] \\
&= \frac{|\mathcal{C}|}{\mathcal{N}}\mathrm{Tr}\left[\rho^n\right]\left(1 + \frac{1}{|\mathcal{C}|}\sum_{\mathsf{b}\neq 1,\mathsf{c}\in\mathcal{C}} \frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}}\,\frac{\mathrm{Tr}\left[\rho^n \mathsf{b}\right]}{\mathrm{Tr}\left[\rho^n\right]}\right),
\end{aligned} \tag{19}$$

because, as commented above, in the limit $\tilde{q} \to 0$,

$$\frac{\mathrm{Tr}\left[\rho^n \mathsf{b}\right]}{\mathrm{Tr}\left[\rho^n\right]} \approx \tilde{q}^{\Delta_{\mathsf{b}}^1/n} \to 0\,, \tag{20}$$

with Δ_{b}^1 the scaling dimension of the vacuum corresponding to the vacuum of the b-twisted sector. Therefore, the quantity

$$\Delta_n[\mathcal{C}] = \frac{1}{|\mathcal{C}|} \sum_{\mathsf{b}\neq 1, \mathsf{c}\in\mathcal{C}} \frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}} \frac{\mathrm{Tr}\left[\rho^n \mathsf{b}\right]}{\mathrm{Tr}\left[\rho^n\right]}, \tag{21}$$

is subleading and only appears in the fusion category case, as in the group like case, b can only be the identity in $N_{\mathsf{a}\bar{\mathsf{a}}}^{\mathsf{b}}$ appearing in (18), that is, $\Delta_n[\mathcal{C}]$ identically vanishes for the group-like case. Now, by writing

$$\mathrm{Tr}\left(\rho_{\mathcal{C}}\rho^{n-1}\right) = \mathrm{Tr}\left[\rho^n\right]\left[\frac{|\mathcal{C}|}{\mathcal{N}}\left(1+\Delta_n[\mathcal{C}]\right)\right], \tag{22}$$

one may express the relative entropy as

$$\begin{aligned} S(\rho_{\mathcal{C}}||\rho) &= \lim_{n\to 1}\frac{1}{n-1}\log\left[\frac{\mathrm{Tr}\left[\rho_{\mathcal{C}}^n\right]}{\mathrm{Tr}\left[\rho^n\right]\left[\frac{|\mathcal{C}|}{\mathcal{N}}\left(1+\Delta_n[\mathcal{C}]\right)\right]}\right] \\ &= \lim_{n\to 1}\frac{1}{n-1}\left(\log\frac{\mathrm{Tr}\left[\rho_{\mathcal{C}}^n\right]}{\mathrm{Tr}\left[\rho^n\right]} - \log\left[\frac{|\mathcal{C}|}{\mathcal{N}}\left(1+\Delta_n[\mathcal{C}]\right)\right]\right) \\ &= \lim_{n\to 1}\left(-\Delta S_n + \frac{1}{1-n}\log\left[\frac{|\mathcal{C}|}{\mathcal{N}}\left(1+\Delta_n[\mathcal{C}]\right)\right]\right). \end{aligned} \tag{23}$$

Noting that

$$\begin{aligned} \mathcal{N} &= \sum_{\mathsf{a},\mathsf{b}\in\mathcal{C}} N_{\mathsf{aa}}^{\mathsf{b}}\mathrm{Tr}[\rho\,\mathsf{b}] = \sum_{\mathsf{a},\mathsf{b}\in\mathcal{C}}\sum_{\mathsf{c}}\frac{S_{\mathsf{ac}}S_{\mathsf{ac}}S_{\mathsf{bc}}}{S_{\mathbb{1}\mathsf{c}}}\mathrm{Tr}[\rho\,\mathsf{b}] \\ &= \sum_{\mathsf{b},\mathsf{c}}\frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}}\mathrm{Tr}[\rho\mathsf{b}] = |\mathcal{C}|\,\mathrm{Tr}[\rho] + \sum_{\mathsf{b}\neq 1,\mathsf{c}}\frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}}\mathrm{Tr}[\rho\mathsf{b}] \\ &= |\mathcal{C}|\,\mathrm{Tr}[\rho]\left(1+\frac{1}{|\mathcal{C}|}\sum_{\mathsf{b}\neq 1,\mathsf{c}}\frac{S_{\mathsf{bc}}}{S_{1\mathsf{c}}}\frac{\mathrm{Tr}[\rho\mathsf{b}]}{\mathrm{Tr}[\rho]}\right) = |\mathcal{C}|\left(1+\Delta_1[\mathcal{C}]\right), \end{aligned} \tag{24}$$

where we have used that, for diagonal RCFTs, $\bar{\mathsf{a}} = \mathsf{a}$ and we have taken the definition of $\Delta_1[\mathcal{C}]$ from (26). One may finally write

$$S(\rho_{\mathcal{C}}||\rho) = \lim_{n\to 1}\left(-\Delta S_n + \frac{1}{1-n}\log\frac{\left(1+\Delta_n[\mathcal{C}]\right)}{\left(1+\Delta_1[\mathcal{C}]\right)}\right), \tag{25}$$

where ΔS_n is given by,

$$\Delta S_n = \frac{1}{1-n}\log\frac{\mathrm{Tr}[\rho_{\mathcal{C}}^n]}{\mathrm{Tr}[\rho^n]}. \tag{26}$$

This quantity, when restricted to a subsystem A of the theory, is known as entanglement asymmetry, quantifying the degree of symmetry breaking in a region of an extended quantum system [14,15].

We note that while for the group-like case, where $\Delta_n[\mathcal{C}] = 0$ exactly vanishes and thus, $S(\rho_{A,\mathcal{C}}||\rho_A) = -\Delta S_1$, for the category case, at leading order in $\tilde{q}\to 0$ we have

$$S(\rho_{\mathcal{C}}||\rho) \sim -\Delta S_1\,. \tag{27}$$

Using the results above, we are now prompted to compute ΔS_n. Therefore, by inserting (11) into (26), one obtains

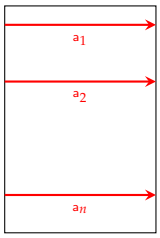
$$\Delta S_n = \frac{1}{1-n}\log\frac{1}{\mathcal{N}^n}\sum_{\mathsf{a}_1,\ldots,\mathsf{a}_n\in\mathcal{C}}\frac{\mathrm{Tr}[\rho\,\bar{\mathsf{a}}_1\mathsf{a}_2\rho\bar{\mathsf{a}}_2\ldots\bar{\mathsf{a}}_{n-1}\mathsf{a}_n\rho\bar{\mathsf{a}}_n\mathsf{a}_1]}{\mathrm{Tr}[\rho^n]}. \tag{28}$$

With this, the multi-charged moments can be defined as

$$Z[q^n,\mathfrak{a}] = \mathrm{Tr}[\rho\,\bar{\mathsf{a}}_1\mathsf{a}_2\rho\bar{\mathsf{a}}_2\ldots\bar{\mathsf{a}}_{n-1}\mathsf{a}_n\rho\bar{\mathsf{a}}_n\mathsf{a}_1] \tag{29}$$

where $\mathfrak{a} \in \mathcal{C}^n$ is the n-tuple defined by $\mathfrak{a} = (\mathsf{a}_1, \ldots, \mathsf{a}_n)$, and graphically reads as

$$Z[q^n, \mathfrak{a}] \equiv Z[q^n, \mathsf{a}_1, \cdots, \mathsf{a}_n] = \tag{30}$$



Given this, now Equation (29) can be written as

$$\Delta S_n = \frac{1}{1-n} \log \frac{1}{\mathcal{N}^n} \sum_{\mathfrak{a} \in \mathcal{C}^n} \frac{Z[q^n, \mathfrak{a}]}{Z[q^n]} \,. \tag{31}$$

As commented above, the TDL $\mathsf{a} \in \mathcal{C}$ are free to move up and down along the (Euclidean) time direction of the replicated manifold under continuous transformations, and this allows to fuse them. Using this again, one can write,

$$Z[q^n, \mathfrak{a}] = \mathrm{Tr}[\rho^n \bar{\mathsf{a}}_1 \mathsf{a}_2 \bar{\mathsf{a}}_2 \ldots \bar{\mathsf{a}}_{n-1} \mathsf{a}_n \bar{\mathsf{a}}_n \mathsf{a}_1] \,. \tag{32}$$

Now, one may use the fusion algebra $\mathsf{a} \times \mathsf{b} = \sum_{\mathsf{c}} N_{\mathsf{ab}}^{\mathsf{c}} \mathsf{c}$ to, assuming that $\bar{\mathsf{a}} = \mathsf{a}, \ \forall \mathsf{a} \in \mathcal{C}$, write,

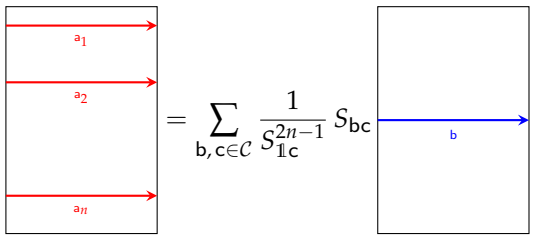
$$\mathsf{a}_1\mathsf{a}_2\mathsf{a}_2\mathsf{a}_3\mathsf{a}_3 \ldots \mathsf{a}_{n-1}\mathsf{a}_{n-1}\mathsf{a}_n\mathsf{a}_n\mathsf{a}_1 = \sum_{\mathsf{b}'s, \mathsf{c}'s} N_{\mathsf{a}_1\mathsf{a}_2}^{\mathsf{b}_1} N_{\mathsf{b}_1\mathsf{a}_2}^{\mathsf{c}_1} N_{\mathsf{c}_1\mathsf{a}_3}^{\mathsf{b}_2} N_{\mathsf{b}_2\mathsf{a}_3}^{\mathsf{c}_2} N_{\mathsf{c}_2\mathsf{a}_4}^{\mathsf{b}_3} N_{\mathsf{b}_3\mathsf{a}_4}^{\mathsf{c}_3} \ldots N_{\mathsf{c}_{n-2}\mathsf{a}_n}^{\mathsf{b}_{n-1}} N_{\mathsf{b}_{n-1}\mathsf{a}_n}^{\mathsf{c}_{n-1}} N_{\mathsf{c}_{n-1}\mathsf{a}_1}^{\mathsf{d}} \, \mathsf{d} \tag{33}$$

Using the Verlinde formula (Equation (3)) and general properties of the S-matrix, we obtain (see Appendix A)

$$\sum_{\mathfrak{a}} Z[q^n, \mathfrak{a}] = \sum_{\mathsf{b} \in \mathcal{C}} \sum_{\mathsf{c} \in \mathcal{C}} \frac{1}{S_{\mathbb{1}\mathsf{c}}^{2n-1}} S_{\mathsf{bc}} \, \mathrm{Tr}[\rho^n \, \mathsf{b}] \,, \tag{34}$$

that can be represented as

$$\sum_{\mathsf{a}_1, \cdots \mathsf{a}_n} = \sum_{\mathsf{b}, \mathsf{c} \in \mathcal{C}} \frac{1}{S_{\mathbb{1}\mathsf{c}}^{2n-1}} S_{\mathsf{bc}} \,, \tag{35}$$



Finally, ΔS_n in (26) can be written as

$$\Delta S_n = \frac{1}{1-n} \log \frac{1}{\mathrm{Tr}[\rho^n]} \frac{\sum_{\mathsf{b},\mathsf{c}} \frac{1}{S_{\mathbb{1}\mathsf{c}}^{2n-1}} S_{\mathsf{bc}} \, \mathrm{Tr}[\rho^n \mathsf{b}]}{\left(\sum_{\mathsf{b},\mathsf{c}} \frac{S_{\mathsf{bc}}}{S_{\mathbb{1}\mathsf{c}}} \mathrm{Tr}[\rho \mathsf{b}] \right)^n} \,. \tag{36}$$

We provide here the asymptotic limit $\tilde{q} \to 0$ for ΔS_n in Equation (36). This amounts to taking only the terms related to $\mathsf{b} = \mathbb{1}$ in the sums above, from which one can derive that

$$\Delta S_n \sim \frac{1}{1-n} \log \frac{1}{\mathrm{Tr}[\rho^n]} \frac{\sum_{\mathsf{c}} \frac{1}{S_{\mathbb{1}c}^{2(n-1)}} \mathrm{Tr}[\rho^n]}{(\sum_{\mathsf{c}} \mathrm{Tr}[\rho \, \mathbb{1}])^n} = \frac{1}{1-n} \log \frac{1}{|\mathcal{C}|^n} \sum_{\mathsf{c} \in \mathcal{C}} S_{\mathbb{1}\mathsf{c}}^{2(1-n)} \,. \tag{37}$$

In limit $n \to 1$ one finds

$$\Delta S_1 = \log |\mathcal{C}| + \frac{2}{|\mathcal{C}|} \sum_{\mathsf{c} \in \mathcal{C}} \log S_{\mathbb{1}\mathsf{c}} \tag{38}$$

As it is said above, it would be expected that the relative entropy is related to the vacuum expectation values of the TDL implementing the symmetry under consideration. To see this, let us recall the definition of quantum dimension,

$$d_{\mathsf{c}} = \frac{S_{\mathbb{1}\mathsf{c}}}{S_{\mathbb{1}\mathbb{1}}} \tag{39}$$

so we can write (38) as

$$\Delta S_1 = \log|\mathcal{C}| + 2\log S_{\mathbb{1}\mathbb{1}} + \frac{2}{|\mathcal{C}|}\sum_{\mathsf{c}\in\mathcal{C}} \log d_{\mathsf{c}} \tag{40}$$

On the other hand

$$1 = \sum_{\mathsf{c}\in\mathcal{C}} S_{\mathbb{1}\mathsf{c}} S_{\mathbb{1}\mathsf{c}} = S_{\mathbb{1}\mathbb{1}}^2 \sum_{\mathsf{c}\in\mathcal{C}} d_{\mathsf{c}}^2 \tag{41}$$

which implies

$$S_{\mathbb{1}\mathbb{1}} = \frac{1}{\mathcal{D}}, \qquad \mathcal{D} = \sqrt{\sum_{\mathsf{c}\in\mathcal{C}} d_{\mathsf{c}}^2} \tag{42}$$

where $\mathcal{D}$ is the total quantum dimension of the modular tensor category. Replacing (42) into (40) gives,

$$\Delta S_1 = \log|\mathcal{C}| - 2\log\mathcal{D} + \frac{2}{|\mathcal{C}|}\sum_{\mathsf{c}\in\mathcal{C}} \log d_{\mathsf{c}}\,. \tag{43}$$

Finally, we express $S(\rho_{\mathcal{C}}||\rho)$ in a more illustrative form by observing that the equation above can be written as

$$-\Delta S_1 = \log\left(\frac{1}{|\mathcal{C}|}\sum_{\mathsf{a}\in\mathcal{C}} d_{\mathsf{a}}^2\right) - \frac{1}{|\mathcal{C}|}\sum_{\mathsf{a}\in\mathcal{C}} \log d_{\mathsf{a}}^2\,, \tag{44}$$

and therefore, according to our result in Equation (26), at leading order for categories, and exactly for groups,

$$S(\rho_{\mathcal{C}}||\rho) = \log\left(\frac{1}{|\mathcal{C}|}\sum_{\mathsf{a}\in\mathcal{C}} d_{\mathsf{a}}^2\right) - \frac{1}{|\mathcal{C}|}\sum_{\mathsf{a}\in\mathcal{C}} \log d_{\mathsf{a}}^2\,, \tag{45}$$

which implies that $S(\rho_{\mathcal{C}}||\rho) \geq 0$ always. To see where this property comes from, we may rewrite the relative entropy as:

$$S(\rho_{\mathcal{C}}||\rho) = \log\frac{\left(\sum_{\mathsf{a}} d_{\mathsf{a}}^2\right)/|\mathcal{C}|}{\sqrt[|\mathcal{C}|]{\left(\prod_{\mathsf{a}} d_{\mathsf{a}}^2\right)}}\,. \tag{46}$$

We notice that this is the quotient between the arithmetic mean and the geometric mean of d_{a}^2. As $d_{\mathsf{a}}^2 \in \mathbb{R}^+$, we can invoke the AM–GM inequality [16], resulting in the desired property $S(\rho_{\mathcal{C}}||\rho) \geq 0$. Moreover, the inequality only saturates if all the elements in the mean values are equal to each other. In our case, we always must have an element satifying $d_{\mathbb{1}} = 1$, i.e., the identity. This means that $S(\rho_{\mathsf{c}}||\rho) = 0$ if and only if $d_{\mathsf{a}} = 1\ \forall \mathsf{a}$, that is, just for group-like symmetries. This result is interesting as it shows that,

1. At leading order, the relative entropy $S(\rho_{\mathcal{C}}||\rho)$ only depends on the expectation values of the TDLs or quantum dimensions.
2. As commented above, for groups, in which every symmetry operator is implemented by a TDL a with $d_{\mathsf{a}} = 1$, $S(\rho_G||\rho) = 0$. As an example, for the $G = \mathbb{Z}_2$ group, we have two invertible TDLs $\{\mathbb{1}, \eta\}$ with $\eta \times \eta = \mathbb{1}$ and,

$$d_{\mathbb{1}} = 1, \quad d_{\eta} = 1 \tag{47}$$

Thus,

$$S(\rho_G||\rho) = -\Delta S_1 = 0\,. \tag{48}$$

3. For systems with non-invertible symmetries represented by a fusion category $\mathcal{C}$, there are some TDL's a with $d_{\mathsf{a}} > 1$, and therefore, $S(\rho_{\mathcal{C}}||\rho) > 0$. As an example, for the Fibonacci category $\mathcal{C}_{\text{Fib}}$ we have two TDLs $\{\mathbb{1}, W\}$ with $W \times W = \mathbb{1} + W$ and,

$$d_{\mathbb{1}} = 1, \quad d_W = \phi \tag{49}$$

where $\phi = (1+\sqrt{5})/2$. Thus,

$$S(\rho_{\mathcal{C}}||\rho) = -\Delta S_1 = 0.111572\ldots \tag{50}$$

4. At leading order, the relative entropy $S(\rho_{\mathcal{C}}||\rho)$ essentially counts the number of invariant configurations of the replica partition functions under the action of TDLs insertions, that is, for how many configurations $\mathfrak{a} \in \mathcal{C}^n$, $Z[q^n, \mathfrak{a}] \sim Z[q^n, \mathbb{1}]$. It has been shown that in spontaneously broken phases, the action of TDLs implementing symmetries on the vacua of the theory may be different depending on the phase [3,4]. In our setting, this would lead to a different counting on the configurations satisfying $Z[q^n, \mathfrak{a}] \sim Z[q^n, \mathbb{1}]$. This suggest that $S(\rho_{\mathcal{C}}||\rho)$ may characterize distinct phases associated to the SSB of categorical-symmetries.

5. Conclusions and Outlook

We have studied the relative entropy between the ground states of RCFTs representing the critical point of phase transitions between different symmetry-broken phases of theories with categorical symmetries and their symmetrized versions.

We find that at leading order, this relative entropy only depends on the expectation values of the TDLs or quantum dimensions of the topological operators implementing symmetry. This dependence is such that, for group-like symmetries in which all the quantum dimensions implementing the symmetry are equal to one, this order parameter vanishes identically. On the other hand, in the case of categorical symmetries, we show that this order parameter is not zero. Our results suggest that our proposal can be used to characterize different broken phases of 2D theories with categorical symmetries.

As said above, as the relative entropy is an experimentally accessible quantity, we note that, in extended quantum systems such as the theories addressed here, taking a measure is concomitant of considering a specific subsystem. In this situation, it is thus sensible to extend our ideas by using tools from the theory of entanglement. The entanglement asymmetry, commented above, has been recently introduced as a measure of symmetry breaking at a subsystem level. It is worth investigating if our entropic order parameter for categorical symmetries can be extended to this framework, following the ideas in [17].

Author Contributions: Conceptualization, J.M.-V., P.S.-B. and G.S.; Validation, J.M.-V., P.S.-B. and G.S.; Formal analysis, J.M.-V., P.S.-B. and G.S.; Investigation, J.M.-V.; Writing—original draft, J.M.-V.; Writing—review & editing, J.M.-V., P.S.-B. and G.S. All authors have read and agreed to the published version of the manuscript.

Funding: The work of P.S.-B. is supported by Fundación Séneca de la Región de Murcia, grant 21609/FPI/21. J.M.-V. thanks the financial support of Spanish Ministerio de Ciencia e Innovación PID2021-125700NAC22. G.S. acknowledges financial support through the Spanish MINECO grant PID2021-127726NB-I00, the CSIC Research Platform on Quantum Technologies PTI-001 and the QUANTUM ENIA project Quantum Spain through the RTRP-Next Generation within the framework of the Digital Spain 2026 Agenda.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Data is contained within the article.

Acknowledgments: We thank Arpit Das for many fruitful discussions on these and related topics.

Conflicts of Interest: The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Appendix A

In this appendix, we show how to derive Equation (34) in the main text. Let us focus momentarily in the case $n = 3$ where

$$\mathsf{a}_1\mathsf{a}_2\mathsf{a}_2\mathsf{a}_3\mathsf{a}_3\mathsf{a}_1 = \sum_{\mathsf{b}_1,\mathsf{b}_2,\mathsf{c}_1,\mathsf{c}_2} N_{\mathsf{a}_1\mathsf{a}_2}^{\mathsf{b}_1} N_{\mathsf{b}_1\mathsf{a}_2}^{\mathsf{c}_1} N_{\mathsf{c}_1\mathsf{a}_3}^{\mathsf{b}_2} N_{\mathsf{b}_2\mathsf{a}_3}^{\mathsf{c}_2} N_{\mathsf{c}_2\mathsf{a}_1}^{\mathsf{d}}\, \mathsf{d} \tag{A1}$$

Using the Verlinde formula

$$N_{ij}^{k} = \sum_{m} \frac{S_{im} S_{jm} S_{km}}{S_{\mathbb{1}m}} \tag{A2}$$

we obtain

$$\sum_{j,k} N_{ij}^{k} N_{kj}^{l} = \sum_{j,k} \sum_{m,n} \frac{S_{im} S_{jm} S_{km}}{S_{\mathbb{1}m}} \frac{S_{kn} S_{jn} S_{ln}}{S_{\mathbb{1}l}} = \sum_{m} \frac{S_{im} S_{lm}}{S_{\mathbb{1}m}^2} \tag{A3}$$

where we used that $\sum_j S_{ij} S_{jk} = \delta_{ik}$ and $S_{ij} = S_{ji}$. The sum over the labels in (A1) gives

$$\begin{aligned}
\sum_{\mathsf{a}_1,\mathsf{a}_2,\mathsf{a}_3} \mathsf{a}_1\mathsf{a}_2\mathsf{a}_2\mathsf{a}_3\mathsf{a}_3\mathsf{a}_1 &= \sum_{\mathsf{a}_1,\mathsf{c}_1,\mathsf{c}_2} \left(\sum_{\mathsf{a}_2,\mathsf{b}_1} N_{\mathsf{a}_1\mathsf{a}_2}^{\mathsf{b}_1} N_{\mathsf{b}_1\mathsf{a}_2}^{\mathsf{c}_1} \right) \left(\sum_{\mathsf{a}_3,\mathsf{b}_2} N_{\mathsf{c}_1\mathsf{a}_3}^{\mathsf{b}_2} N_{\mathsf{b}_2\mathsf{a}_3}^{\mathsf{c}_2} \right) N_{\mathsf{c}_2\mathsf{a}_1}^{\mathsf{d}}\, \mathsf{d} \\
&= \sum_{\mathsf{a}_1,\mathsf{c}_1,\mathsf{c}_2} \sum_{m_1,m_2} \frac{S_{\mathsf{a}_1 m_1} S_{\mathsf{c}_1 m_1}}{S_{\mathbb{1}m_1}^2} \frac{S_{\mathsf{c}_1 m_2} S_{\mathsf{c}_2 m_2}}{S_{\mathbb{1}m_2}^2} N_{\mathsf{c}_2\mathsf{a}_1}^{\mathsf{d}}\, \mathsf{d} \\
&= \sum_{\mathsf{a}_1,\mathsf{c}_1,\mathsf{c}_2} \sum_{m_1,m_2,m_3} \frac{S_{\mathsf{a}_1 m_1} S_{\mathsf{c}_1 m_1}}{S_{\mathbb{1}m_1}^2} \frac{S_{\mathsf{c}_1 m_2} S_{\mathsf{c}_2 m_2}}{S_{\mathbb{1}m_2}^2} \frac{S_{\mathsf{c}_2 m_3} S_{\mathsf{a}_1 m_3} S_{\mathsf{d} m_3}}{S_{\mathbb{1}m_3}}\, \mathsf{d} \\
&= \sum_{m_1,m_2,m_3} \frac{\delta_{m_1 m_2} \delta_{m_2 m_3} \delta_{m_3 m_1} S_{\mathsf{d} m_3}}{S_{\mathbb{1}m_1}^2 S_{\mathbb{1}m_2}^2 S_{\mathbb{1}m_3}}\, \mathsf{d} = \sum_{m} \frac{S_{\mathsf{d}m}}{S_{\mathbb{1}m}^5}\, \mathsf{d}
\end{aligned} \tag{A4}$$

For arbitrary n it is easy to find the general results which reads as,

$$\sum_{\mathsf{a}_1,\ldots,\mathsf{a}_n} \mathsf{a}_1\mathsf{a}_2\mathsf{a}_2 \ldots \mathsf{a}_{n-1}\mathsf{a}_n\mathsf{a}_n\mathsf{a}_1 = \sum_{m} \frac{S_{\mathsf{d}m}}{S_{\mathbb{1}m}^{2n-1}}\, \mathsf{d}\,, \tag{A5}$$

and hece

$$\sum_{\mathfrak{a}} Z[q^n, \mathfrak{a}] = \sum_{\mathsf{d}} \sum_{m} \frac{S_{\mathsf{d}m}}{S_{\mathbb{1}m}^{2n-1}} \mathrm{Tr}(\rho^n \mathsf{d})\,. \tag{A6}$$

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Chapter 6

Conclusions

This text was devoted to the study of symmetries and aspects related to them using different techniques. In particular, the main subjects of study of the thesis were generalized symmetries, anomalies, anomalous transport phenomena and non-invertible or categorical symmetries. We have reviewed this aspects in quantum theories, we have mainly studied the chiral fermion theory in 4-dimensions and 2-dimensional conformal field theories, where we have computed entropic quantities as tools to characterize and study the structure of categorical symmetries. The thesis was structured as a compendium of four papers where this subjects were studied. Now, we will present a summary of the conclusions and future prospects for those articles.

Starting with symmetries and anomalies, we have studied the chiral anomaly and transport phenomena associated to it. In that research, we have established a connection between the chiral anomaly in massless fermions quantum field theory and anomalous transport currents at finite temperature and chemical potential in 4-dimensions. For that, we have integrated the anomaly equations in black hole background. By performing direct integration in this background and setting vanishing boundary conditions for the currents in the horizon, we are able to obtain the anomalous coefficients of the chiral magnetic and chiral vortical effects.

Though direct integration can apparently be applied to every theory with a chiral anomaly, it does not work for spin 3/2 fields. [106] This is due to the axial current in that theory not being invariant under fermionic gauge transformations, therefore not being a good local observable. New theories for relativistic spin 3/2 fields have been introduced to solve this [107, 108], where anomalous transport properties and relation to direct integration have been studied [109, 110].

It is also worth to mention the gravitational anomaly in spin 1 gauge fields [111, 112], where the helicity current can be integrated to reproduce the CVE despite the current not being a gauge invariant operator [113]. For these reasons, an interesting future line is the study of anomaly integration for the higher bosonic and fermionic Zilch currents [114, 115, 116].

As explained in the main text, the chiral anomaly can be rewritten as a non-invertible operator. For this reason, the study of categorical symmetries may be of interest for the study of the chiral anomaly. Categorical symmetries are mainly

found in 2-dimensional quantum theories, where superposition makes them possible to exist. In this theories, quantum non-local correlations are characterized by the entanglement entropy. In our works we have studied modified versions of entanglement entropy including non-invertible symmetries to characterize them.

One of the entropic quantities we were interested in is the symmetry resolved entanglement entropy. We firstly proposed the symmetry resolution of entanglement for non-invertible symmetries restricted to the untwisted Hilbert space of the theory. There we found fundamental the study of boundary conditions on the entangling points, being deeply related to the categorical symmetry being anomalous.

In a second study, we provided the full symmetry resolution for the entanglement entropy in 2-dimensional rational CFTs. To do so, we studied generalized charged sectors of the fusion category given by irreps of the tube algebra, or equivalently, by anyons in the 3-dimensional SymTFT. Thus, defining anyonic Cat-SREE. In this case, we also studied the relations between the boundary conditions and the categorical symmetry being anomalous and obtained the general result for symmetry resolution for anyonic sectors. From this result we are able to recover the group-like symmetry resolution as a special case. By studying the structure of the SREE, we realized that entanglement equipartition is broken even if the symmetry is abelian. This characteristic is unique to categorical symmetries, thus recovering equipartition for the ordinary symmetry case.

These works have drawn the attention of the community as some other references studying non-invertible symmetry resolution have appeared. In particular it is worth mentioning the works [92, 117, 118], where symmetry resolution involving strip algebras was introduced. In this works, symmetry resolution is studied for conformal field theories with boundaries, for which the use of strip algebras is more suitable than tube algebras. Though the conclusion reached looks similar to our works in some cases, the setup used is really different, making both approaches relevant in different physical situations. The approach taken in the thesis presents the symmetry resolution with respect to generalized charges of a system described by a CFT, meanwhile the approach of [92, 117, 118] is relevant when considering systems where physical boundaries are present. We believe both approaches are complementary and they explore interesting features of categorical-symmetry resolution in different scenarios.

Our two works presented the first approach to symmetry resolution including categorical symmetries. In the setup of the SymTFT, symmetry resolution can be studied for more general cases than just restricting to diagonal rational CFTs. Some interesting scenarios might be non-diagonal RCFTs [119] and non-unitary theories. Another interesting future direction of this work is computing symmetry resolution in the lattice with non-invertible operators [120, 121, 122, 123] and connecting with the results obtained for the CFTs in the continuum limit.

Finally, the approach we have followed for the computation of entanglement entropy in this cases is the so-called worldsheet approach, consisting on working with the manifold where the theory is defined. A parallel approach consist on interpreting the replica trick as a replication of the field content. This is known as the target space approach [77, 78, 85]. The target space approach involves the definition of the branch-point twist fields, and reduces the computation of entanglement entropy to a conformal correlator. The extension of this approach to include non-invertible symmetries has not yet be explored. This is a line being considered for an upcoming work.

One of the key applications of symmetries is the study of phases of matter. In the context of the Landau paradigm, phases of matter are characterized by different spontaneous symmetry breaking patterns of a global symmetry. Phase transitions are then driven by local order parameters. Some entropic quantities allow to differentiate between different phases of matter. These are the so-called entropic order parameters, counting with examples like the relative entropy and the entanglement asymmetry.

In the last paper, we studied the relative entropy between a vacuum state and its symmetrized version including categorical symmetries. For that, we proposed a symmetrization map including non-invertible symmetries. The map proposed did not satisfy some of the desirable properties of a symmetrizing map. This would be fixed later by [102]. Despite that, we could obtain that, at leading order, relative entropy only depends on the quantum dimensions of the non-invertible symmetry operators, being non-zero just for non-invertible operators.

This work presented a first approach to the study of entropic order parameters for non-invertible symmetries. This study has been significantly extended by recent papers [101, 102, 103, 104]. In these works, alternative versions of symmetrizing maps are presented, allowing for a full computation of an entropic order parameter in QFT. This entropic order parameter can be applied to the characterization of quantum phases such as SPT phases, which leaves the door open to very promising research lines in the field.

In this line, higher-form group-like symmetries have also been studied using similar techniques in 3-dimensional theories [105]. Following these ideas, possibilities for the study of different phases of categorical symmetries in more than 2 dimensions open.

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