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Advances in the Study of Topological
Properties of Analytic Flows on Surfaces

Avances en el Estudio de Propiedades
Topológicas de Flujos Analíticos sobre Superficies

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# ADVANCES IN THE STUDY OF TOPOLOGICAL PROPERTIES OF ANALYTIC FLOWS ON SURFACES

# AVANCES EN EL ESTUDIO DE PROPIEDADES TOPOLÓGICAS DE FLUJOS ANALÍTICOS SOBRE SUPERFICIES

José Ginés Espín Buendía

2017

Quiero empezar dejando por escrito mi más sincero agradecimiento al director de esta Tesis Doctoral, el profesor Víctor Jiménez López. Es justo reconocer que este trabajo es casi más suyo que mío: sin su apoyo, científico y sobre todo personal, esta Tesis Doctoral no hubiera visto nunca la luz. Él sabe bien que el respeto académico, científico y personal que le profeso es imposible de plasmar sobre el papel. No puedo imaginar a ninguna otra persona que hubiera sido capaz de orquestar mi quehacer investigador, o que me hubiera podido entender, aconsejar y guiar en lo personal, de forma tan magnánima como lo ha hecho Víctor. Gracias.

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Debes intentar vislumbrar como un todo una pequeñisima
parte de un todo infinitamente mayor. A pesar de ello,
nunca nadie podrá saberlo todo de esa pequeñísima.
parte. Siempre debes estar preparado y tener alerta
tu capacidad para sorprenderte porque alguien
o algo lo acabará haciendo.

(José Julio Buendía Buendía, Decano de la segunda generación.)

# Contents

# Resumen

A grandes rasgos, este trabajo se propone investigar el efecto de la analiticidad en el campo de los sistemas dinámicos continuos en dimensión 2, es decir, qué fenómenos dinámicos aparecen cuando la función que define un sistema de este tipo es analítica. Más concretamente, se trata de avanzar en la investigación de la naturaleza topológica de los flujos analíticos sobre superficies en los siguientes cuatro ámbitos:

1. la clasificación topológica de los atractores globales inestables para flujos polinómicos en el plano;
2. la caracterización topológica de los conjuntos $\omega$-límite para flujos analíticos en abiertos de la esfera y el plano proyectivo;
3. la caracterización topológica de los conjuntos periódicos límite para familias de flujos polinómicos en el plano;
4. el estudio de los flujos analíticos en superficies con todas sus órbitas densas.

En lo que sigue, sin pretensión alguna de ser escrupulosos en el rigor y la formalidad, resumimos de forma somera lo que entendemos por cada uno de los cuatro puntos previos, remarcando cuál ha sido nuestra contribución en cada caso.

Empezamos introduciendo algo de notación y algunas definiciones.

Llamaremos *superficie* a todo espacio topológico conexo, Hausdorff y segundo axioma de numerabilidad, $S$, tal que cada punto $p \in S$ posee un entorno homeomorfo a un subconjunto abierto y conexo de $\mathbb{R}^2$. Un *flujo* en una superficie $S$ es una aplicación continua $\Phi : \mathbb{R} \times S \to S$ tal que, para cada $p \in S$ y cada $t, s \in \mathbb{R}$, $\Phi(0, p) = p$ y $\Phi(t + s, p) = \Phi(t, \Phi(s, p))$. A veces, por abreviar, escribiremos simplemente que $(S, \Phi)$ es un flujo, entendiendo implícitamente que $S$ es una superficie y que $\Phi$ es un flujo en $S$.

Dado un flujo $(S, \Phi)$ y un punto $p \in S$, decimos que $\varphi_\Phi(p) = \Phi(\mathbb{R} \times \{p\})$ (y también que su parametrización $t \mapsto \Phi_p(t) = \Phi(t, p)$) es la *órbita de* $p$. Cuando la órbita $\varphi_\Phi(p)$ es un conjunto unipuntual, decimos que $p$ (o $\varphi_\Phi(p)$) es *singular*; un punto que no es singular se dice que es *regular*. Si $\varphi_\Phi(p)$ es una circunferencia topológica (esto es, un subconjunto de $S$ homeomorfo a la circunferencia unidad $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 =$

1}), decimos que $\varphi(p)$ es una *órbita periódica*. Si una órbita $\varphi_\Phi(p)$ no es ni singular ni periódica, su parametrización $\Phi_p : \mathbb{R} \to S$ es un embebimiento (una aplicación continua e inyectiva). Notemos que cualesquiera dos órbitas o bien coinciden o bien son disjuntas: el flujo folia a la superficie $S$ en una unión de circunferencias topológicas, puntos singulares y embebimientos de la recta real en $S$. Por otro lado, asociado a cualquier órbita podemos considerar una dirección temporal para la misma (la dada por su parametrización por $\Phi$); cuando $S$ se ve como la unión de sus órbitas cada una con su dirección temporal, decimos que $S$ es el *retrato de fases* del flujo.

Estamos principalmente interesados en los flujos que no son solo continuos sino también analíticos. Una aplicación $f : O \to \mathbb{R}$ ($y = f(x)$ con $x = (x_1, \ldots, x_n)$), definida en un abierto $O$ de $\mathbb{R}^n$, se dice que es *analítica* si, para cada $x_0 = (x_{0,1}, \ldots, x_{0,n}) \in O$, $f$ se puede representar como una serie de potencias convergente en las variables $x_1, \ldots, x_n$ en algún entorno de $x_0$. Una función vectorial $f : O \subset \mathbb{R}^n \to \mathbb{R}^m$ se llama analítica si cada una de sus componentes es una aplicación analítica en el sentido anterior. La noción de analiticidad se puede extender de forma natural a funciones definidas en superficies — presentaremos más abajo la definición formal de lo que entendemos por superficie analítica y por función analtíca sobre una superficie analítica (véase Section 1.4.1).

El concepto de flujo sobre superficies esta directamente relacionado con el de ecuaciones diferenciales autónomas. Para ilustrar este hecho, centrémonos por un momento en el caso $S = \mathbb{R}^2$. Supongamos que $f : \mathbb{R}^2 \to \mathbb{R}^2$ es una función vectorial analítica y consideremos, para cada $z_0 \in \mathbb{R}^2$, la única solución maximal $t \mapsto \Phi_{z_0}(t)$ de la ecuación autónoma $\dot{z}(t) = f(z(t))$ verificando $z(0) = z_0$. Si todas las órbitas están definidas en toda la recta real, la función $\Phi(t, z) = \Phi_z(t)$ es un flujo analítico en $\mathbb{R}^2$. Recíprocamente, si $\Phi$ es un flujo analítico en $\mathbb{R}^2$ y definimos $f(z) = \frac{\partial \Phi}{\partial t}(0, z)$, entonces para cada $z_0 \in \mathbb{R}^2$, $t \mapsto \Phi_{z_0}(t)$ es la solución maximal $z(t)$ de la ecuación $\dot{z}(t) = f(z(t))$ verificando $z(0) = z_0$.

En general, la solución maximal de una ecuación diferencial no tiene por qué estar definida en todo $\mathbb{R}$. Sin embargo, como veremos más abajo en el cuerpo de la tesis, siempre podemos modificar cualquier ecuación diferencial autónoma dada para conseguir una segunda ecuación que tiene exactamente el mismo conjunto de imágenes de soluciones pero con todas las nuevas soluciones definidas en la recta real. En este trabajo, la mayor parte de las veces vamos a estar exclusivamente preocupados por las propiedades topológicas de los conjuntos de órbitas y el "cambio" anterior se puede hacer sin alterar el "contexto topológico". Para simplificar nuestra discusión en este resumen introductorio, asumiremos que, cada vez que nos refiramos a una ecuación diferencial autónoma, todas las soluciones están definidas en toda la recta real.

Nuestro primer resultado destacable en este trabajo concierne al estudio de la estructura de los conjuntos de ceros de funciones analíticas (definidas en abiertos del plano euclídeo). Dado un abierto $O \subset \mathbb{R}^n$, decimos que el conjunto $A \subset O$ es un *conjunto analítico* (en $O$) si $A$ es el conjunto de ceros de alguna función analítica con $O$ como dominio, esto es, si existe una función analítica $f : O \to \mathbb{R}$ tal que $A = \{x \in O : f(x) = 0\}$. Como tendremos ocasión de comprobar a lo largo de la tesis, conocer la estructura topológica de los conjuntos analíticos planos (esto es, de los conjuntos analíticos en abiertos de $\mathbb{R}^2$) es de tremenda importancia cuando se trabaja con dinámica topológica de flujos analíticos bidimensionales.

En 1971, Dennis Sullivan probó que, salvo en algunos casos triviales, un conjunto analítico plano es localmente una unión finita de un número par de arcos que dos a dos se cortan en un solo punto (un extremo común para cada uno de los arcos), esto es, un conjunto homeomorfo a $\{z \in \mathbb{C} : z^m \in [0,1]\}$ para algún $m$ — véase [79].

Para describir la estructura topológica de los conjuntos analíticos planos, se puede proceder en dos etapas: primero se prueba la estructura local de estrella, sin importar la paridad (esto es el conocido como el Teorema de Lojasiewicz [52, Theorem 6.3.3, p. 168]); posteriormente se prueba que el número de ramas de la estrella es par (Teorema de Sullivan). El Teorema de Lojasiewicz es un corolario de dos resultados clásicos de analiticidad real: el Teorema de preparación de Weierstrass y el Lema de Hensel. Sus demostraciones, aunque algo tediosas (especialmente la del lema de Hensel), son elementales, véase [52]. Por contra, todas las pruebas del Teorema de Sullivan que conocíamos, incluida la original, requieren de herramientas avanzadas de la topológica algebraica [15, 16, 32, 41], y por ende son difíciles de seguir para el lector no especialista en la materia. En [25], en colaboración con V. Jiménez, presentamos una nueva prueba, de ambos pasos en la discusión anterior, basada en argumentos elementales y bien conocidos de la teoría cualitativa de las ecuaciones diferenciales ordinarias en el plano. En el capítulo 2 exponemos con detalle esta prueba.

Como en otros campos de las matemáticas, uno de los problemas más importantes en el campo de la *teoría cualitativa de las ecuaciones diferenciables* es el que concierne a la "clasificación" de los sistemas. Dar una "clasificación completa" de todos los flujos que se pueden definir sobre una superficie concreta es una tarea de extrema dificultad. En particular, clasificar completamente los flujos polinómicos en el plano, esto es, aquellos flujos asociados a sistemas de la forma

$$\begin{cases} x' = P(x,y), \\ y' = Q(x,y), \end{cases}$$

con $P(x,y)$ y $Q(x,y)$ polinomios en las variables $x$ y $y$, es un problema clásico (quizás el problema por excelencia de la teoría cualitativa de las ecuaciones diferenciables) que, de cerrarse, daría, como subproducto, respuesta a la segunda parte del famoso *problema dieciséis de Hilbert* (en el que se pregunta por la existencia y, en su caso su cómputo, de una cota $H(n)$ para el número máximo de ciclos límite que un sistema polinómico puede tener en función del máximo de los grados $n$ de los polinomios $P(x,y)$ y $Q(x,y)$ — por ciclo límite se entiende a cualquier órbita periódica aislada de otras órbitas periódicas en el diagrama de fases. Hasta el presente, esta cota es desconocida incluso para el caso $n = 2$; de hecho, aunque se conjetura que $H(2) = 4$, ni se ha podido probar la finitud de $H(2)$.

En consecuencia, los investigadores en el área han ido añadiendo restricciones dinámicas y/o analíticas para abordar el problema. Por cierto, "clasificar" significa aquí que dos flujos son "lo mismo" si son topológicamente equivalentes. (Dos flujos $(S_1, \Phi_1)$ y $(S_1, \Phi)$ se dicen *topológicamente equivalentes* cuando existe un homeomorfismo $h : S_1 \to S_2$ llevando órbitas a órbitas y preservando la dirección temporal.)

Hasta donde nosotros conocemos, la clasificación topológica de los sistemas polinómicos presentando un punto singular como atractor global, esto es, los sistemas cuyas órbitas tienden todas en tiempo positivo a un mismo punto singular, no se ha estudiado aún. En el capítulo 4 atacamos este problema. Este contenido se corresponde con el artículo [28], en colaboración con V. Jiménez. Con esto cubrimos el primero de los puntos en la lista de arriba.

Para resolver el anterior problema, en particular para encontrar buenas condiciones que garanticen que dos sistemas con un punto singular atractor global sean topológicamente equivalentes, nos vimos en la necesidad de aplicar el Teorema de Markus-Neumann (véase el Teorema 3.2 abajo). En términos informales, este resultado dice que dos flujos en una superficie $S$ son topológicalmente equivalentes si, y solo si, existe un homeomorfismo de $S$ en $S$ preservando órbitas y direcciones de un conjunto particular de órbitas destacadas de ambos flujos (las conocidas como *configuraciones por separatrices*). Intentando ajustar este resultado a nuestro contexto de trabajo, encontramos de forma inesperada varios contraejemplos. En el capítulo 3 discutimos algunos de estos contratiempos y presentamos una versión correcta y generalizada del teorema. Este capítulo está basado en el artículo [26], que también se hizo en colaboración con V. Jiménez.

El capítulo 5 trata del estudio del segundo de los puntos en la lista de arriba. Se trabaja allí con la noción de los conjuntos $\omega$-límite. Dado un flujo $(S, \Phi)$ y un punto $p \in S$, definimos el conjunto $\omega$-límite de $p$ como

$$\omega_\Phi(p) = \{q \in S : \exists t_n \to \infty; \Phi_p(t_n) \to q\}.$$

Posiblemente, el resultado más conocido de la teoría cualitativa de las ecuaciones diferenciales sea el Teorema de Poincaré-Bendixson.

*Teorema* (Teorema de Poincaré-Bendixson). Consideremos un flujo de clase $C^1$, $\Phi$, en $\mathbb{R}^2$ y un punto $p \in \mathbb{R}^2$ cuya órbita está acotada (es decir, $\varphi_\Phi(p)$ está contenida en algún compacto de $\mathbb{R}^2$). Entonces o bien $\omega_\Phi(p)$ contiene algún punto singular o bien $\omega_\Phi(p)$ es él mismo una órbita periódica.

Ña importancia de este resultado está justificada no solo por su significado como herramienta en el campo de la teoría cualitativa de las ecuaciones diferenciales sino también por la cantidad de matemáticas que ha motivado; [18] es un buen tratado sobre el tema.

H. Poincaré fue el primero en demostrar el resultado para el caso de los flujos analíticos, más tarde I. Bendixson lo extendió al caso de flujos de clase $C^1$. Destaquemos que el teorema describe los conjuntos $\omega$-límite de una órbita acotada que no contienen puntos singulares en dos sentidos: desde un punto de vista dinámico, dice que $\omega_\Phi(p)$ es una órbita periódica; desde el punto de vista topológico, dice que $\omega_\Phi(p)$ es una circunferencia topológica. Por otra lado, también es importante remarcar que el resultado está lejos de presentar una caracterización de todos los conjuntos $\omega$-límite: solo da restricciones para una subfamilia de estos conjuntos. En esta dirección, y hasta donde nosotros sabemos, fue R. E. Vingroad el primero en dar una caracterización topológica global de los conjuntos $\omega$-límite asociados a los flujos (continuos) en la esfera.

*Teorema* (Vinograd, [82]). Sea $\Phi$ un flujo en la esfera euclidea $\mathbb{S}^2$ y $p \in \mathbb{S}^2$. Entonces $\omega_\Phi(p)$ es la frontera de un conjunto conexo no vacío y propio $O \subset \mathbb{S}^2$ tal que $\mathbb{S}^2 \setminus O$ también es conexo.

Recíprocamente, si $\Omega$ es la frontera de un conjunto conexo no vacío y propio $O \subset \mathbb{S}^2$ con complementario $\mathbb{S}^2 \setminus O$ conexo, entonces existe un flujo $(\Phi, \mathbb{S}^2)$ y un punto $p \in \mathbb{S}^2$ tal que $\omega_\Phi(p) = \Omega$.

En [43], V. Jiménez y J. Llibre presentaron caracterizaciones topológicas de los conjuntos $\omega$-límite para los flujos analíticos en el plano, la esfera y el plano proyectivo (y en sus abiertos). Por ejemplo, en el caso de la esfera prueban que:

*Teorema* (Llibre-Jiménez). Sea $\Phi$ un flujo analítico en $\mathbb{S}^2$ y $p \in \mathbb{S}^2$. Entonces $\omega_\Phi(p)$ es o bien un conjunto unipuntual (un punto singular) o bien la frontera de un cactus en la esfera (por cactus entendemos una unión finita y simplemente conexa de discos).

Recíprocamente, para cada cactus $A \subset \mathbb{S}^2$, hay un flujo analítico $\Phi$ en $\mathbb{S}^2$ y un homeomorfismo $h : \mathbb{S}^2 \to \mathbb{S}^2$ tal que $h(\operatorname{Bd} A)$ es un conjunto $\omega$-límite para $\Phi$.

En [43], se distinguen dos etapas. Primeramente, se caracterizan los conjunto límites para flujos definidos en todo el plano, toda la esfera y todo el plano proyectivo; seguidamente se extienden estos resultados a los abiertos de estas superficies. En el capítulo 5 revisaremos estas caracterizaciones y veremos que, mientras que los resultados enunciados en la primera de las dos anteriores etapas son correctos, las caracterizaciones presentadas en la segunda de las etapas no corren igual suerte. Más aún, ambas caracterizaciones (incluyendo aquellas dadas para flujos definidos en todo el plano, toda la esfera o todo el plano proyectivo) están basadas en un lema auxiliar que hemos probado ser incorrecto. Sin ánimo de ser pulcros en rigor aquí, este resultado auxiliar viene a decir que en cualquier superficie analítica, todo flujo analítico tiene la siguiente propiedad: si una órbita visita ambos *lados* de un arco de puntos singulares contenido en su conjunto $\omega$-límite, el flujo debe tener la misma orientación en ambos lados. En la sección 5.1, mostraremos algunos contraejemplos a este lema auxiliar (y probaremos que las demostraciones de caracterizaciones enunciadas en [43] se pueden corregir para el caso del plano, de la esfera y del plano proyectivo); por su parte, en la sección 5.2, lidiaremos con el problema de caracterizar los conjuntos $\omega$-límite para flujos analíticos en abiertos cualesquiera de la esfera. Estas dos secciones se corresponden, respectivamente, con los artículos [30] y [27], ambos en colaboración con V. Jiménez. Destaquemos que, en el caso de los abiertos de la esfera, hemos sido capaces de encontrar restricciones que debe verificar un subconjunto de la esfera para poder ser el conjunto $\omega$-límite de algún flujo analítico en un abierto de la esfera dado (véase el Teorema F). Creemos que estas restricciones son suficientes para caracterizar topológicamente esos conjuntos límite: en este sentido, en la sección 5.2 conjeturamos lo que creemos que es el recíproco del Teorema F y damos ya un primer paso para su prueba (Proposición 5.24). La caracterización de los conjuntos $\omega$-límite para flujos analíticos en abiertos del plano proyectivo (y en general en superficies arbitrarias) es todavía un problema abierto que esperamos poder atacar a corto y medio plazo.

El tercero de los puntos en la lista que presentamos al inicio de este resumen versa sobre el concepto de los conjuntos periódicos límite. Dada una familia $\{\Phi_\lambda\}_{\lambda\in\mathbb{R}}$ de flujos polinómicos planos, decimos que $\Gamma \subset \mathbb{S}^2$ es un conjunto periódico límite para $\{\Phi_\lambda\}_\lambda$ si existen sucesiones $(\lambda_n)_n$ en $\mathbb{R}$ y $(p_n)_n$ en $\mathbb{R}^2$ tales que, para cada $n$, la órbita $\varphi_{\Phi_{\lambda_n}}(p_n)$ es un ciclo límite para $\Phi_{\lambda_n}$ y la sucesión de circunferencias $(\varphi_{\Phi_{\lambda_n}}(p_n))_n$ converge (en la métrica de Hausdorff) a $\Gamma$.

El estudio de la estructura de los conjuntos periódicos límite para los flujos polinómicos planos es recurrente en la teoría de bifurcaciones y en el tratamiento del problema dieciséis de Hilbert (véase [72]). Por ejemplo, el programa propuesto por F. Dumortier, R. Roussarie y C. Rousseau para resolver la parte existencial del problema dieciséis de Hilbert para sistemas polinómicos cuadráticos se divide en el análisis de 121 casos basados en esta noción de conjunto periódico límite.

En el capítulo 6, basado en un trabajo en colaboración con A. Belotto (véase [11]), presentamos una caracterización topológica de todos los conjuntos periódicos límites para familias de flujos polinómicos planos.

Finalmente, en el capítulo 7 abordamos el estudio del cuarto y último de los puntos listados al principio del resumen. Este capítulo recoge una colaboración con D. Peralta-Salas y G. Soler donde estudiamos las superficies admitiendo flujos analíticos teniendo a todas sus órbitas como conjuntos densos (véase [29]). Estas superficies se llaman *minimales*. Las superficies minimales orientables quedaron totalmente caracterizadas por J.C. Benière in 1998, dejando abierto el caso no orientable. El capítulo 7 cierra la discusión para el caso de superficies de género finito. También se construye un ejemplo de una superficie minimal de género infinito y se conjetura que cualquier superficie no orientable de género infinito es minimal.

# Summary

In broad terms, this dissertation intends to investigate the effect of analyticity in the field of continuous dynamical systems of dimension 2, that is, what dynamical phenomena appear when the function defining such a system is analytic. Being more concrete, we aim to advance in the study of the topological nature of the analytic flows on surfaces in the following four areas:

1. the topological classification of unstable global attractors for polynomials flows on the plane;
2. the topological characterization of the $\omega$-limit sets for analytic flows on open subsets of the sphere and the projective plane;
3. the topological characterization of the limit periodic sets for families of polynomial flows on the plane;
4. the study of the analytic flows on surfaces with the property of having all their orbits dense.

In what follows we summarize what we mean by each of the previous four points, highlighting what our contribution in each case has been.

We first start introducing some needed notation and definitions.

By a *surface* we mean a connected, second countable, Hausdorff space $S$ such that every point $p \in S$ possesses a neighbourhood homeomorphic to a connected open set of $\mathbb{R}^2$. A *flow* on a surface $S$ is a continuous map $\Phi : \mathbb{R} \times S \to S$ such that, for every $p \in S$ and every $t, s \in \mathbb{R}$, $\Phi(0, p) = p$ and $\Phi(t + s, p) = \Phi(t, \Phi(s, p))$. Sometimes, for short, we will simple write that $(S, \Phi)$ is a flow, understanding implicitly that $S$ is a surface and that $\Phi$ is a flow on $S$.

Given a flow $\Phi$ on a surface $S$ and a point $p \in S$, we say that $\varphi_\Phi(p) = \Phi(\mathbb{R} \times \{p\})$ (and also its parametrization $t \mapsto \Phi_p(t) = \Phi(t, p)$) is the *orbit through* $p$. When the orbit $\varphi_\Phi(p)$ is a singleton, we say that $p$ (or $\varphi_\Phi(p)$) is *singular*; when a point is not singular, it is said to be *regular*. If $\varphi_\Phi(p)$ is a topological circle (that is, a subset of $S$ homeomorphic to the euclidean unit circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$), then we say that $\varphi_\Phi(p)$ is a *periodic orbit.* If an orbit $\varphi_\Phi(p)$ is neither singular nor periodic, then its parametrization $\Phi_p : \mathbb{R} \to S$ is an embedding (a continuous inyective map). Notice that any two orbits either coincide or do not meet so the surface $S$ is foliated as union of circles, singular points and injective continuous images of the real line. On the other hand, any orbit has an associated time direction (that one given by its parametrization under $\Phi$); when $S$ is seen as the union of its orbits with their time directions, we say that $S$ is the *phase portrait* of the flow.

We are mainly interested in flows which are not only continuous but also analytic. A map $f : O \to \mathbb{R}$ ($y = f(x)$ with $x = (x_1, \dots, x_n)$), defined on an open set $O$ of $\mathbb{R}^n$ is said to be *analytic* if, for every $x_0 = (x_{0,1}, \dots, x_{0,n}) \in O$, $f$ can be represented as a convergent power series in the variables $x_1, \dots, x_n$ in some neighbourhood of $x_0$. A vector map $f : O \subset R^n \to \mathbb{R}^m$ is called analytic if each of its components is analytic in the previous sense. The notion of analyticity can be extend to maps defined on analytic surfaces — the formal definition for what we understand by analytic surface and by analytic map on an analtytic surface will be presented below (see Section 1.4.1).

The concept of flow over a surface is directly related to that of autonomous differential equation. To illustrate this fact, let us focus on the case $S = \mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an analytic vector map and consider, for every $z_0 \in \mathbb{R}^2$, the only (maximal) solution $\Phi_{z_0}(t)$ of the autonomous equation $\dot{z}(t) = f(z(t))$ such that $z(0) = z_0$. If all of those solutions are defined on the whole real line, then the function $\Phi(t, z) = \Phi_z(t)$ is an analytic flow on $\mathbb{R}^2$. Reciprocally, if $\Phi$ is an analytic flow on $\mathbb{R}^2$ and we define $f(z) = \frac{\partial \Phi}{\partial t}(0, z)$, then for every $z_0 \in \mathbb{R}^2$, $t \mapsto \Phi_{z_0}(t)$ is the maximal solution $z(t)$ of the equation $\dot{z}(t) = f(z(t))$ satisfying $z(0) = z_0$.

In general, the maximal solution of a differential equation needs not be defined on the whole $\mathbb{R}$. Nevertheless, as we will show later in the dissertation, given any autonomous differential equation, we can always modify the equation to get a second one having exactly the same set of images of solutions but with all of those solutions defined on the whole real line. In most of the cases, we are only interested in the topological properties of the set of orbits and the previous "change" can be done without altering the "topological context". To simplify our discussion in this introductory abstract, we will always assume that, when referring to a differential system, all solution are defined on the whole real line.

Our first remarkable result in the thesis deals with the structure of the set of zeros of analytic maps (on open subsets of the euclidean plane). Given an open set $O \subset \mathbb{R}^n$, we say that a subset $A \subset O$ is an *analytic set* (in $O$) if $A$ is the set of zeros of some analytic map with $O$ as domain, that is, if there exists an analytic map $f : O \to \mathbb{R}$ such that $A = \{x \in O : f(x) = 0\}$. As will transpire along this thesis, knowing the local topological structure of planar analytic sets (that is, analytic sets in open subsets of $\mathbb{R}^2$) is of paramount importance when dealing with topological dynamics of bidimensional analytic flows.

In 1971, Dennis Sullivan proved that any planar analytic set is locally, up to some trivial cases, a finite union of evenly many arcs which pairwise meet at an only point (a common endpoint for each arc); that is, a set homeomorphic to $\{z \in \mathbb{C} : z^m \in [0,1]\}$ for some $m$ — see [79].

To describe the topological structure of planar analytic sets, one can proceed in two steps: first, the local star structure is proved (this is the so-called Lojasiewicz Theorem [52, Theorem 6.3.3, p. 168]); then one shows (Sullivan's Theorem) that the number of branches is even. Lojasiewicz's Theorem is a corollary of two classical results on real analyticity: the Weierstrass preparation theorem and the Hensel lemma. Their proofs, if somewhat cumbersome (especially in the case of Hensel's lemma), are elementary, see [52]. In contrast, all proofs of Sullivan's Theorem we are aware of, including the original one, require advanced tools of algebraic topology [15, 16, 32, 41], and hence are hard to follow for the non-specifically trained reader. In [25], in collaboration with V. Jiménez, we presented a simple and dynamically based proof of both steps. Chapter 2 is devoted to the exposition of that proof.

As in other fields of mathematics, one of the most important problems in the field of *continuous dynamical systems* is that concerning the "classification" of the systems. Finding a "complete classification" for all flows in a particular surface is an exceedingly difficult task. In particular, classifying completely the plane polynomial flows, that is, those flows associated with systems of the form

$$\begin{cases} x' = P(x,y), \\ y' = Q(x,y), \end{cases}$$

with $P(x,y)$ and $Q(x,y)$ polynomials in the variables $x$ and $y$, is a classical problem (many would say the problem *par excellence*) of the *qualitative theory of differential equations* which, if completed, would provide, as a by-product, an answer to the famous (second part of the) Hilbert 16th problem asking for a bound $H(n)$ on the number of limit cycles of the system in terms of the maximum degree $n$ of $P(x,y)$ and $Q(x,y)$ — by a limit cycle we mean a periodic orbit which is isolated from other periodic orbits. Presently, this bound is unknown even in the quadratic case $n = 2$; in fact, although there are strong reasons to conjecture $H(2) = 4$, not even the finiteness of $H(2)$ has been established.

Understandably, researchers in this area have added dynamical and/or analytic restrictions to try and tackle the problem. In this context, by the way, "classifying" means that two flows are "the same" if they are topologically equivalent. (Two flows $(S_1, \Phi_1)$ and $(S_1, \Phi)$ will be said to be *topologically equivalent* when there exists a homeomorphism $h : S_1 \to S_2$ taking orbits onto orbits and preserving the time directions.)

As far as we know, the problem of classifying, up to topological equivalence, the polynomial systems with a globally attracting singular point, that is, those whose orbits tend in positive time to the same singular point, has not been studied yet. In Chapter 4 we dealt with this problem. This piece of work corresponds with the paper [28], in collaboration with V. Jiménez. With this we cover the first of the items in the list above.

To solve the previous problem, in particular to find good conditions guaranteeing that two given systems with a globally attracting singular point are topologically equivalent we need to apply the Markus-Neumann Theorem (see Theorem 3.2 below for a rigorous statement). Roughly speaking, this result says that two flows on a surface $S$ are topologically equivalent if and only if there is a homeomorphism of $S$ onto itself preserving the orbits and time directions of particular sets of orbits from both flows (the so-called separatrix configurations). While trying to apply this result to our context of work, we unexpectedly found some counterexamples to it. In Chapter 3 we discuss those counterexamples and present a corrected (and generalized) version of the theorem. The contents of this chapter are based on the paper [26], also in collaboration with V. Jiménez.

Chapter 5 is dedicated to the study of the second item in the list above. It deals with the notion of $\omega$-limit sets. Given a flow $(S, \Phi)$ and a point $p \in S$ we define the $\omega$-limit set of $p$ as

$$\omega_\Phi(p) = \{q \in S : \exists t_n \to \infty; \Phi_p(t_n) \to q\}.$$

Arguably, the most celebrated and well-knwon result in the field of qualitative theory of differential equations is the Poincaré-Bendixson Theorem.

*Theorem* (Poincaré-Bendixson Theorem). Let $\Phi$ be a $C^1$ flow on $\mathbb{R}^2$ and $p \in \mathbb{R}^2$ be a point whose orbit is bounded (i.e. $\varphi_\Phi(p)$ is contained in some compact subset of $\mathbb{R}^2$). Then either $\omega_\Phi(p)$ contains some singular point or $\omega_\Phi(p)$ is itself a periodic orbit.

The importance of this result has to do not only with its usefulness as a tool in the field of the qualitative theory of plane differential equations but also with the amount of mathematics which has motivated; [18] is a good survey on the topic.

The theorem was first proved by Poincaré for the case of analytic flows, later Bendixson extended the result to the $C^1$ case. We remark that the theorem describes the $\omega$-limit set of a bounded orbit whose $\omega$-limit set contains no singular points in two senses: from a dynamical point of view, it says that $\omega_\Phi(p)$ is a periodic orbit; from a topological point of view, it says that $\omega_\Phi(p)$ is homeomorphic to the euclidean unit circle $\mathbb{S}^1$. On the other hand, it is also important to highlight that the result is far from being a characterization of $\omega$-limit sets: it only gives constrains about a subfamily of $\omega$-limit sets. It this direction, and as far as we know, R. E. Vingroad was the first one being able to give a global topological characterization for $\omega$-limit sets associated with (continuous) flows on the sphere:

*Theorem* (Vinograd [82]). Let $\Phi$ be a flow on the euclidean sphere $\mathbb{S}^2$ and let $p \in \mathbb{S}^2$. Then $\omega_\Phi(p)$ is the boundary of a nonempty proper open connected subset $O \subset \mathbb{S}^2$ such that $\mathbb{S}^2 \setminus O$ is also connected.

Reciprocally, if $\Omega$ is the boundary of a nonempty proper open connected subset $O \subset \mathbb{S}^2$ such that $\mathbb{S}^2 \setminus O$ is also connected, then there exists a (continuous) flow $(\Phi, \mathbb{S}^2)$ and a point $p \in \mathbb{S}^2$ such that $\omega_\Phi(p) = \Omega$.

In [43], V. Jiménez and J. Llibre presented characterizations, up to homeomorphism, of the $\omega$-limit sets for analytic flows on the plane, the sphere and the projective plane (and their open subsets). For example, for the case of the sphere they found that:

*Theorem* (Llibre-Jiménez). Let $\Phi$ be an analytic flow on $\mathbb{S}^2$ and let $p \in \mathbb{S}^2$. Then $\omega_\Phi(p)$ is either a singleton (a singular point) or the boundary of a cactus in the sphere (here a cactus means a finite simply connected union of disks).

Conversely, for every cactus $A \subset \mathbb{S}^2$, there are an analytic flow $\Phi$ on $\mathbb{S}^2$ and a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $h(\operatorname{Bd} A)$ is the $\omega$-limit set for some orbit of $\Phi$.

In [43], the authors distinguished two different stages. Firstly, they characterized those limit sets for flows defined on the whole sphere, the whole plane and the whole projective plane; afterwards, they extended the results to open subsets of these surfaces. In Chapter 5 we will check those characterizations and see that, while the results given for flows on the whole three surfaces are correct, the characterization for their open subsets need to be redone. Even more, both characterizations (including those for the whole surfaces) are based on an incorrect auxiliary result which says, roughly speaking, that on any general analytic surface all analytic flows have the following property (stated here without intention of being rigorous): if an orbit meets both sides of an arc of singular points contained in its $\omega$-limit set, then the flow must be equally oriented in both sides. In Section 5.1 we show some counterexamples to that auxiliary lemma (and show how the prove of the characterizations in [43] can be amended for the cases of the plane, the sphere and the projective plane); then, in Section 5.2, we deal with the problem of characterizing $\omega$-limit sets for analytic flows on general open subsets of the sphere. This two sections correspond, respectively, with the articles [30] and [27], both in collaboration with V. Jiménez. We remark that, in the case of the open subsets of the sphere, we have been able to find some restrictions which a subset of a given open subset of the sphere must satisfy to be an $\omega$-limit set for some analytic flow on the given open set (see Theorem F). We do believe that these restrictions are enough to characterize, from a topological point of view, these limit sets: in this sense, in Section 5.2, we conjecture what we believe is a converse of Theorem F and we already give a first step of its proof (Proposition 5.24). The characterization of the $\omega$-limit sets for analytic flows on proper open subsets of the projective plane (and, in general, on general surfaces) is still an open problem.

The third of the items listed above refers to the concept of limit periodic sets. Given a family $\{\Phi_\lambda\}_{\lambda \in \mathbb{R}}$ of plane polynomial flows, we say that a set $\Gamma \subset \mathbb{S}^2$ is a limit periodic set for $\{\Phi_\lambda\}_\lambda$ if there exist sequences $(\lambda_n)_n$ in $\mathbb{R}$ and $(p_n)_n$ in $\mathbb{R}^2$ such that, for every $n$, the orbit $\varphi_{\Phi_{\lambda_n}}(p_n)$ is a limit cycle for $\Phi_{\lambda_n}$ and the sequence of circles $(\varphi_{\Phi_{\lambda_n}}(p_n))_n$ converges (in the Hausdorff topology) to $\Gamma$.

The study of the structure of the limit periodic sets of planar polynomial flows is a central object in bifurcation theory and in the treatment of the Hilbert $16^{th}$ problem (see R. Roussarie's book [72]). For example, the program of F. Dumortier, R. Roussarie and C. Rousseau [23] to solve the existential part of the $16^{th}$ Hilbert problem for quadratic polynomial systems requires the analysis of 121 cases based on the limit periodic sets.

In Chapter 6, which is based on a collaboration with A. Belotto (see [11]), we present a topological characterization for all limit periodic sets of families of plane polynomial flows.

Finally, the aim of Chapter 7 is to study the fourth, and last, item of the list above. That chapter collects a collaboration with D. Peralta-Salas and G. Soler where analytic minimal surfaces are taken into consideration (see [29]).

Surfaces admitting flows with every orbit being dense are called *minimal*. Minimal orientable surfaces were characterized by J.C. Benière in 1998, leaving open the nonorientable case. Chapter 7 fills this gap in the finite genus case: a characterization of analytic minimal nonorientable surfaces of finite genus is given. We also construct an example of a minimal nonorientable surface with infinite genus and conjecture that any nonorientable surface is minimal.

# Chapter 1

# Basic results and notions on analytic flows on surfaces

The aim of this first chapter is to introduce some of the more basic notions which we will be using during the rest of the dissertation. Among others, we have used as main references [22, 43, 52, 54, 55] and [66].

## 1.1 Some basic notation and topological notions

A number of standard topological notions will be of repeated use along the thesis; we collect in this section some of them. Good references containing all the more basic notions not included here are, for example, [53] and [62].

Given a topological space $X$ and a subset $A \subset X$, we will write $\mathrm{Int}_X(A)$, $\mathrm{Cl}_X(A)$ and $\mathrm{Bd}_X(A)$ to denote, respectively, the interior, the closure and the boundary of $A$ in $X$; if there is no possible confusion, we will omit the sub-index $X$ to simply write $\mathrm{Int}(A)$, $\mathrm{Cl}(A)$ and $\mathrm{Bd}(A)$.

We say that a topological space is an *arc* (respectively, *open arc*, *circle*, *disk*, an *open disk*, a *sphere*) if it is homeomorphic to $[0,1]$ (respectively, $\mathbb{R}$, the unit circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$, the unit disk $\mathbb{D}^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$, the real plane $\mathbb{R}^2$, the euclidean sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$). Sometimes, we will add the adjective "topological" to clarify we are using the concept up to homeomorphism. If $L$ is a (topological) arc, and $h : [0,1] \to L$ is a homeomorphism, then $h(0)$ and $h(1)$ are called the *endpoints* of $L$.

In general, neighbourhoods will not be considered to be open sets. Given a topological space $X$, a point $p \in X$ (respectively a set $P \subset X$) and $A$ a subset of $X$, we will say that $A$ is a *neighbourhood of* $p$ (respectively of $P$) if there exists an open set $B$ containing $p$ (respectively of $P$) and contained in $A$. A point $p \in X$ is said to be *isolated* (in $X$) if $\{p\}$ is a neighbourhood of $p$, that is, if $\{p\}$ is an open subset of $X$. By a *discrete set* we mean a subset $Y \subset X$ consisting of isolated points.

### 1.1.1 Some notions on connectedness

A topological space is said to be *locally connected at a point* $p$ if every open neighbourhood of $p$ contains a connected neighbourhood of $p$; the space is *locally connected* if it is locally connected at every point.

We say that $X$ is *pathwise connected* if for any $x, y \in X$ there is a continuous map $\varphi : [0,1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$. Such a map (as well as its range $\varphi(I)$, if no confusion arises) is called a *path* (from $x$ to $y$). If additionally, for any $x, y \in X$, there is an arc in $X$ having $x$ and $y$ as its endpoints, then $X$ is called *arcwise connected*. When $X$ is Hausdorff, these turn out to be equivalent notions, see [85, Corollary 31.6, p. 222].

A compact connected Hausdorff space is called a *continuum*; if a subspace of a continuum $X$, with the induced topology, is itself a continuum, then it is called a *subcontinuum* of $X$. A locally connected metric continuum is called a *Peano space*. The Hahn-Mazurkiewicz theorem establishes that a (nonempty) continuum is a Peano space if and only if it is the continuous image of the interval $[0,1]$ [54, Theorem 2, p. 256]. Hence any Peano space is pathwise (arcwise) connected.

Given a topological space $X$ and a continuous map $F : [0,1] \times [0,1] \to X$, we will say that $F$ is a *homotopy* between the paths $\varphi, \varphi' : [0,1] \to X$ given by $\varphi(t) = F(t,0)$ and $\varphi'(t) = F(t,1)$ for every $t$; the paths $\varphi$ and $\varphi'$ are also said to be *homotopic*.

A pathwise connected space $X$ whose fundamental group is trivial (that is, any path $\varphi : [0,1] \to X$ with $\varphi(0) = \varphi(1)$ is homotopic to the constant path $\varphi' : [0,1] \to X$ given by $\varphi'(t) = \varphi(0) = \varphi(1)$ for any $t$) is called *simply connected*. As shown in [36, Proposition 3.2, p. 10]), simply connectedness is equivalent to contractibility: $X$ is said to be *contractible* if there are $p \in X$ and a continuous map $G : [0,1] \times X \to X$ such that $G(0,x) = x$ and $G(1,x) = p$ for any $x$. It is well known (see, e.g, [73, Theorem 13.11, p. 274]) that if $\emptyset \subsetneq X \subsetneq \mathbb{S}^2$ is a region (that is, an open connected set), then $X$ is simply connected if and only $\mathbb{S}^2 \setminus X$ is connected and if and only if $X$ is an open disk. The equivalence between simply connectedness of $X$ and connectedness of $\mathbb{S}^2 \setminus X$ holds as well when $X \subset \mathbb{S}^2$ is a Peano space, see [47, Proposition 4.1].

A set of a topological space is said to be *closed-open* if it is open and closed at the same time. Let $X$ be a topological space and $A$ and $B$ two subsets. Then $X$ is *connected* *between $A$ and $B$* if there is no closed-open set $F$ such that $A \subset F$ and $F \cap B = \emptyset$; $X$ is said to be *totally disconnected* if the space is not connected between any pair of singleton subsets.

The last definition is taken from [54]. We have noticed that other authors define a topological space as totally disconnected if the space has no connected subsets apart from the singleton subsets and the empty set. It is then natural to ask ourselves if both notions coincide. In order to fix our notation we are going to keep the term "totally disconnected" to refer to the spaces as defined in the paragraph above; on the other hand, and following again [54], we reserve the term *hereditarily disconnected* to the latter notion.

In general every totally disconnected space is also hereditarily disconnected. Indeed, suppose that a totally disconnected space $X$ possesses a connected subset $C \subset X$ which is neither the empty set not a singleton subset. Take two different points in $C$, say $p$ and $q$. Since $C$ is connected its only closed-open subsets are itself and the empty set so $C$ (and therefore $X$) is connected between $p$ and $q$. However, there exist hereditarily disconnected spaces which are not totally disconnected (see [54, Remark (ii), p. 152]). Both notions coincide in the context of compact metric spaces (see [54, p. 189]).

### 1.1.2 Some notions on metric spaces

The euclidean spaces $\mathbb{R}^n$ with the standard euclidean distance will play a prominent role along the dissertation. Unless explicitly stated, if we refer to any notion related to a distance on $\mathbb{R}^n$, we will always understand that such a distance is the euclidean one. For example, if we talk about an open ball of center $p \in \mathbb{R}^n$ and radius $r$ we will be meaning the set $B(p,r) := \{q \in \mathbb{R}^n : \|q-p\| < r\}$ where by $\|p\|$ we denote the euclidean norm of $p$; that is, if $p := (p_1, \ldots, p_n)$, $\|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$. We will use $\mathbf{0}$ to refer in short to the origin of $\mathbb{R}^n$.

In general, if $(X,d)$ is a metric space, we will write $B_d(a,r)$ to denote the *open ball* (for $d$) of center $a$ and radius $r$; that is, $B_d(a,r) := \{b \in X : d(a,b) < r\}$. A subset of $X$ is then said to be *bounded* (for $d$) if it is contained in some open ball (for $d$). Given any two subsets $A, B \subset X$, we define the *distance between $A$ and $B$* as the real number $d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}$ and the *diameter of $A$* as $\operatorname{diam}(A) := \sup\{d(a,b) : a,b \in A\}$. Notice that, while $d(A,B)$ equals always a real number, $\operatorname{diam}(A)$ is only a well-defined real number when $A$ is bounded; otherwise we write $\operatorname{diam}(A) = \infty$.

We close this subsection presenting a characterization of the topological spaces which are metrizable (we recall that a topological space $X$, with $\mathcal{T}$ as its set of open subsets, is called *metrizable* when it is possible to define a distance on $X$ such that the open sets generated by the topology on $X$ associated with that distance are exactly the sets in $\mathcal{T}$).

A topological space $X$ is *regular* if for every point $p \in X$ and every closed subset $C \subset X$ there exist disjoint open subsets $U, V \subset X$ such that $p \in U$ and $C \subset V$. A family of subsets of a topological space $X$ is said to be *locally finite* if every point of $X$ possesses a neighbourhood which only meets finitely many subsets of the family.

**Theorem 1.1** (Nagata-Smirnov metrization Theorem). *A topological space is metrizable if and only if is it is regular and has a basis which can be written as the countable union of locally finite collections.*

*Proof.* See, for example, [62, Theorem 40.3, p. 250]. □

### 1.1.3 One-point compactifications

A topological space $X$ is said to be *locally compact at a point* $p \in X$ if $p$ has a compact neighbourhood in $X$; $X$ is called *locally compact* if it is locally compact at all its points.

The proof of the following classic result can be found, for example, in [54, Theorem 5, p. 43].

**Theorem 1.2** (Alexandrov's one-point compactification Theorem). *Every locally compact metric space $X$ is homeomorphic to a subset $Y$ of a compact metric space $X_\infty$ such that $X_\infty \setminus Y$ consists of a single point.*

Given a set $X_\infty$ as in the statement, we will say that $X_\infty$ is *the* one-point compactification of $X$. Notice that we are here committing a small abuse of notation: one-point compactification are not unequivocally defined but they are unique up to homeomorphism (if $X_\infty$ and $X^*_\infty$ are two compact metric spaces as in the previous statement, then it is direct to check that they must be homeomorphic).

**Example 1.3.** The one-point compactification of the euclidean plane $\mathbb{R}^2_\infty = \mathbb{R}^2 \cup \{\infty\}$ is homeomorphic to the euclidean sphere $\mathbb{S}^2$ (see the details in Section 1.4.3 below) — we will also refer to $\mathbb{R}^2_\infty$ (and to $\mathbb{S}^2$) as the *Bendixson compactification* of $\mathbb{R}^2$.

### 1.1.4 Homeomorphisms on the sphere

#### Homeomorphic copies of the sphere

**Proposition 1.4.** *Let $\{C_i\}_i$ be a family of pairwise disjoint continua in $\mathbb{S}^2$. Assume that $\mathbb{S}^2 \setminus C_i$ is connected for any $i$ and, additionally, that one of the following conditions is satisfied:*

*(i) There is a region $O$ such that $\{C_i\}_i$ is the family of connected components of $\mathbb{S}^2 \setminus O$.*

*(ii) The family $\{C_i\}_i$ is countable and (if infinite) the diameters of the sets $C_i$ tend to zero.*

*Then, after defining the equivalence relation $\sim$ in $\mathbb{S}^2$ by $x \sim y$ if either $x = y$ or there is $i$ such that both $x$ and $y$ belong to $C_i$, the quotient space $\Sigma := \mathbb{S}^2/\sim$ is homeomorphic to $\mathbb{S}^2$.*

*Proof.* Let $\Pi : \mathbb{S}^2 \to \Sigma$ be the projection map, when recall that $\mathcal{U}$ is open in $\Sigma$ if and only if $\Pi^{-1}(\mathcal{U})$ is open in $\mathbb{S}^2$. In view of [54, Theorem 8, p. 533] we are left to show:

(*) $\Sigma$ is Hausdorff;

(**) $\Sigma \setminus \{X\}$ is connected for any $X \in \Sigma$.

(***) $\Pi^{-1}(\mathcal{C})$ is connected for any connected set $\mathcal{C} \subset \Sigma$.

To prove (*) we assume first that (i) holds. Let $X, Y \in \Sigma$, $X \neq Y$. We must find disjoint open neighbourhoods $\mathcal{U}(X)$ and $\mathcal{U}(Y)$ of $X$ and $Y$ in $\Sigma$. If $X$ and/or $Y$ is a point from $O$ this is trivial because $O$ is open, so assume that both $X$ and $Y$ are components of $\mathbb{S}^2 \setminus O$. Since $O$ is open [54, Theorem 2, p. 169] implies that $X$ it is the intersection of all open closed sets of $\mathbb{S}^2 \setminus O$ (with respect to the topology of $\mathbb{S}^2 \setminus O$) including it. In particular, it is possible to find disjoint compact sets $A, B$ with $A \cup B = \mathbb{S}^2 \setminus O$, $X \subset A$, $Y \cap B \neq \emptyset$. The connectedness of any set $C_i$ implies that either $C_i \subset A$ or $C_i \subset B$. Therefore, $Y \subset B$. Find pairwise disjoint open sets $V \supset A$ and $W \supset B$. Since, for any $i$, either $C_i \subset V$ or $C_i \subset W$, we get that $\mathcal{U}(X) = \Pi(V)$ and $\mathcal{U}(Y) = \Pi(W)$ are the neighbourhoods we are looking for.

Now we prove (*) assuming that (ii) holds. Given $X, Y \in \Sigma$, $X \neq Y$, we first find disjoint open sets $V, W$ in $\mathbb{S}^2$ with $X \subset V$, $X \subset W$. Realize that the resultant set $V'$ after removing from $V$ the points from the components $C_i$ such that $C_i \cap \operatorname{Bd} V \neq \emptyset$ is also open (here we need that the diameters of the sets $C_i$ go to zero), and the same is true for the analogously defined set $W'$. Then $\mathcal{U}(X) = \Pi(V')$ and $\mathcal{U}(Y) = \Pi(W')$ are disjoint open neighbourhoods of $X$ and $Y$ in $\Sigma$.

Statement (**) is immediate: since $\mathbb{S}^2 \setminus X$ is connected by hypothesis, and $\Pi$ is continuous, $\Pi(\mathbb{S}^2 \setminus X) = \Sigma \setminus \{X\}$ is connected as well.

Note finally that $\Pi$ is a closed map by (*). Then (***) follows from [54, Theorem 9, p. 131] and the fact that any $X \in \Sigma$ is a connected subset of $\mathbb{S}^2$. $\square$

### Extending homeomorphism from subsets to the whole sphere

The study of conditions under which a homeomorphism between two subsets of a manifold $M$ can be extended to a homeomorphism of $M$ onto itself has a long tradition (see references in [4, 51]). A well-known example is the so-called *Schoenflies Theorem* which states that any homeomorphism between two circles in $\mathbb{R}^2$ can be extended to a homeomorphism of $\mathbb{R}^2$ onto itself (see for example [54, Theorem 1, p. 535]). Many authors have treated the particular case of the plane and the sphere: [1, 3, 2, 34, 35] and [54, Section 61.V].

For the purposes of this dissertation, we are mainly interested in the case of Peano spaces in the euclidean sphere. In [3], necessary and sufficient conditions ensuring that a homeomorphism $h$ between two Peano spaces $A_1, A_2 \subset \mathbb{S}^2$ can be extended to a homeomorphism $H : \mathbb{S}^2 \to \mathbb{S}^2$ are given. The final part of this subsection is devoted to present such characterization.

We call a triplet $(A, B, C)$ of arcs in $\mathbb{R}^2_\infty$ sharing a common endpoint $p$ (and no other point) a *triod*. The point $p$ is called the *vertex* of the triod, the other endpoints of the arcs $A, B, C$ being called its *endpoints*. We say that the triod $(A, B, C)$ is *positive*, when, after taking an open euclidean ball $U$ of center $p$ and radius $\epsilon > 0$ small enough, there is $\theta_0 \in \mathbb{R}$ such that the first intersection points of these arcs with $\operatorname{Bd} U$ can be written as $p + \epsilon e^{\mathbf{i}\theta_A}, p + \epsilon e^{\mathbf{i}\theta_B}, p + \epsilon e^{\mathbf{i}\theta_C}$, with $\theta_0 = \theta_A < \theta_B < \theta_C < \theta_0 + 2\pi$. We say that the triod is *negative* when it is not positive. Observe that the definition above excludes the case when the common endpoint $p$ is $\infty$. We then say that $(A, B, C)$ is positive when $(G(A), G(B), G(C))$ is negative, $G : \mathbb{R}^2_\infty \to \mathbb{R}^2_\infty$ being defined by $G(z) = 1/z$ (here we identify $\mathbb{R}^2$ with $\mathbb{C}$ and mean $1/\infty = \mathbf{0}$, $1/\mathbf{0} = \infty$). If $C \subset \mathbb{R}^2$ is a circle around $\mathbf{0}$ and $(q, q', q'')$ is a triplet of distinct points in $C$, then we call it *positive* or *negative* according to whether it is counterclockwise or clockwise oriented in $C$, that is, there is a positive (negative) triod $(A, A', A'')$ in the disk enclosed by $C$ with vertex $\mathbf{0}$ and endpoints $q, q', q''$.

Let $P, P' \subset \mathbb{R}^2$ (respectively, $P, P' \subset \mathbb{R}^2_\infty$). We say that $P$ and $P'$ are $\mathbb{R}^2$-*compatible* (respectively, $\mathbb{R}^2_\infty$-*compatible*) if there is a homeomorphism $H$ from $\mathbb{R}^2$ (respectively, from $\mathbb{R}^2_\infty$) onto itself mapping $P$ onto $P'$. Clearly, $\mathbb{R}^2$-homeomorphisms amount to $\mathbb{R}^2_\infty$-homeomorphisms mapping $\infty$ to itself. If $H : \mathbb{R}^2_\infty \to \mathbb{R}^2_\infty$ is a homeomorphism, then, as it is well known, either it preserves the orientation, that is, all pairs of triods $(A, B, C)$ and $(H(A), H(B), H(C))$ have the same sign, or it reverses the orientation, that is, all pairs of triods $(A, B, C)$ and $(H(A), H(B), H(C))$ have opposite sign. As it turns out, see [3], this is the key property to identify compatibility:

**Theorem 1.5.** *Two Peano sets $P$ and $P'$ in $\mathbb{R}^2_\infty$ are $\mathbb{R}^2_\infty$-compatible if and only if there is a homeomorphism $h : P \to P'$ either preserving or reversing the orientation, in the former sense, for all pair of triods $(A, B, C)$ and $(h(A), h(B), h(C))$ in $P$ and $P'$ (when $h$ can indeed be homeomorphically extended to the whole $\mathbb{R}^2_\infty$).*

The former result can be adapted to the $\mathbb{R}^2$-setting as follows. We say that $P \subset \mathbb{R}^2$ is *nice* if it is unbounded, $P_\infty = P \cup \{\infty\}$ is a Peano subset of $\mathbb{R}^2_\infty$, and for any triod $(A, B, C)$ in $P_\infty$ with vertex $\infty$ there is a $\theta$-curve in $P_\infty$ including $A$, $B$ and $C$ (by a $\theta$-*curve* we mean a union of three arcs intersecting exactly at their endpoints). Then we get: two nice sets $P, P'$ are $\mathbb{R}^2$-compatible if and only if there is a homeomorphism

$h : P \to P'$ either preserving or reversing the orientation for all pair of triods $(A, B, C)$ and $(h(A), h(B), h(C))$ in $P$ and $P'$ (when, again, $h$ can indeed be homeomorphically extended to the whole $\mathbb{R}^2$).

## 1.2 Analytic Functions: definition and basic results

We will denote by $\mathbb{N}$ and $\mathbb{N}_*$ the sets of positive and non-negative integers, respectively. Given any $n \in \mathbb{N}$ (respectively $n = \infty$) we will write $\mathbb{N}_n$ to denote the set $\{1, 2, \ldots, n\}$ (respectively $\mathbb{N}$); sometimes, in order to unify the notation, we will also put $\infty = \infty + 1 = \infty - 1$ so in particular $\mathbb{N}_\infty = \mathbb{N}_{\infty+1} = \mathbb{N}_{\infty-1} = \mathbb{N}$.

Given any $n \in \mathbb{N}$, any $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ and any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_*^n$, we write $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ and $|z|^\alpha = |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n}$.

A *real power series in $n$ real variables centered at $z_0 \in \mathbb{R}^n$* is a formal expression of the type

$$\sum_{\alpha \in \mathbb{N}_*^n} a_\alpha (z - z_0)^\alpha, \tag{1.1}$$

with $a_\alpha \in \mathbb{R}$. We say that the series (1.1) is *absolutely convergent* at $z \in \mathbb{R}^n$ if for a bijection $\phi : \mathbb{N} \to \mathbb{N}_*^n$ the series $\sum_{n=1}^\infty \left|a_{\phi(k)}\right| |z - z_0|^{\phi(k)}$ is convergent (as a numerical series). Recall that if a numerical series converges absolutely then it also converges unconditionally; hence, if the series (1.1) is absolutely convergent, then the numerical series $\sum_{n=1}^\infty a_{\phi'(k)} (z - z_0)^{\phi'(k)}$ converges for any bijection $\phi' : \mathbb{N} \to \mathbb{N}_*^n$ and its sum does not depend on the rearrangement. Thus, if a power series like (1.1) converges absolutely at a point $z$, then we can naturally speak about its *sum* at that point.

Analytic functions are those which can be expressed as a power series around any point in their domain.

**Definition 1.6.** Let $f : U \to \mathbb{R}$ be a function defined on an open subset $U \subset \mathbb{R}^n$. The function $f$ is said to be *analytic* at a point $z_0 \in U$ if there exist an open neighbourhood of $z_0$, $V \subset U$, and a sequence of real numbers $(a_\alpha)_{\alpha \in \mathbb{N}^n}$, such that the power series $\sum_{\alpha \in \mathbb{N}_*^n} a_\alpha (z - z_0)^\alpha$ is absolutely convergent at any $z \in V$ and its sum coincides with $f(z)$. We say that $f$ is *analytic (on $U$)* if it is analytic at any point of $U$; we will also say that $f$ is of class $C^\omega$ (on $U$) or that $f \in C^\omega(U)$.

Functions $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ are described as vectors $f = (f_1, f_2, \ldots, f_m)$, with each $f_i : U \to \mathbb{R}$ being a real function. Given any of the components functions $f_i$, we will write $\frac{\partial f_i}{\partial x_j}(p)$ to denote, when it exists, the partial derivative of $f_i$ with respect to $x_j$ (for $1 \le j \le n$). In general, given any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_*^n$ and any $p \in U$, we will write

$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and, when it makes sense,

$$D_\alpha f_i(p) = \frac{\partial^{|\alpha|} f_i}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}(p).$$

When $\alpha = \mathbf{0}$, the previous expression will simply denote the value of $f_i(p)$. Then, we will say that the function $f_i$ is of class $C^r$, for some $r \in \mathbb{N}$ (respectively for $r = \infty$), if for every $\alpha \in \mathbb{N}_*^n$ with $|\alpha| \leq r$ (respectively for every $\alpha \in \mathbb{N}^n$), the derivative $D_\alpha f_i(p)$ exists and is continuous for every $p \in U$. Finally, we say that $f$ is *analytic* or of class $C^\omega$ (respectively of class $C^r$ for some $0 \leq r \leq \infty$) if all its components $f_i$ are analytic (respectively of class $C^r$) according to the above definition.

Recall that given any of the indexes $1 \leq i \leq n$, if $f_i$ is of class $C^1$, we say that the vector map $\nabla f_i : U \to \mathbb{R}^n$ given by $\nabla f_i(p) = \left(\frac{\partial f_i}{\partial x_1}(p), \ldots, \frac{\partial f_i}{\partial x_n}(p)\right)$ for every $p \in U$ is *the gradient of $f$*.

Analytic functions behave well under algebraic operations: the sum, the product, the division and the composition of analytic functions, when well-defined, are analytic. The same can be said about their calculus: any analytic function is continuously differentiable and any partial derivative of first order at any point of its domain can be computed differentiating formally each term of its representation as an absolutely convergent power series, and we obtain again an absolutely convergent power series. In particular, any analytic function is of class $C^\infty$. We refer the reader to [52] for the details.

Two important and elementary properties of analytic functions are presented below; we will used them repeatedly in the sequel. The first of them works for analytic functions defined on any open subset of $\mathbb{R}^n$, the second one is true only in the one-dimensional case.

**Proposition 1.7.** *If $f : U \to \mathbb{R}$ is an analytic function, $U \subset \mathbb{R}^n$ is a region and $f$ vanishes at an open subset $V$ of $U$, then it vanishes at the whole $U$.*

*Proof.* It follows, after using a standard connectedness argument, from the relation between the coefficients of a power series representing an analytic function and its partial derivatives, see [52, Remark 2.2.4]. □

**Proposition 1.8.** *Let $f : I \to \mathbb{R}$ be an analytic function defined on an open interval $I \subset \mathbb{R}$. If $f$ vanishes at a sequence of points accumulating in $I$, then $f$ vanishes at the whole interval $I$.*

*Proof.* This is a direct application of Rolle's Theorem, see [52, pp. 11–14]. □

The following result is, while still elementary, of an altogether different calibre. It relies on the fact that the ring of local convergent power series is a unique factorization domain; a detailed proof can be found, for instance, in [43, Appendix A].

**Theorem 1.9.** *Let $f = (f_1, f_2) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic function and $w \in U$ be a zero of $f$. Then there are an open neighbourhood of $w$, $W \subset U$, and analytic functions $k, h_1, h_2 : W \to \mathbb{R}$ such that:*

*(i)* $f_1 = kh_1$ *and* $f_2 = kh_2$ *on* $W$*;*

*(ii)* $h = (h_1, h_2)$ *has no zeros in* $W \smallsetminus \{w\}$.

We finally highlight a result which will allow to extend analytic maps defined on open sets of $\mathbb{R}^n$ to the whole $\mathbb{R}^n$ still retaining $C^\infty$ regularity.

### 1.2.1 Extension of analytic maps

**Theorem 1.10.** *Let $U \subset \mathbb{R}^n$ be open. Given any analytic map $F : U \to \mathbb{R}$, there exists a $C^\infty$ map $\rho : \mathbb{R}^n \to (0, \infty)$, which is analytic at $U$ and vanishes outside $U$, such that $\rho F$ (after being extended as zero outside $U$) is $C^\infty$ in the whole $\mathbb{R}^n$.*

The last result is a consequence of [83, Lemma 6]: we devote the rest of the section to justify this statement is true.

We start rewriting [83, Lemma 6] with our notation:

**Lemma 1.11.** *Let $U \subset \mathbb{R}^n$ be an open set and $\{U_m\}_{m \in \mathbb{N}_*}$ be an open cover of $U$ (in general, given a topological space $X$ we say that a family of subsets of $X$ covers $X$ if the union of those subsets equals $X$) such that for every $m \in \mathbb{N}_*$, $\mathrm{Cl}(U_m)$ is a compact subset of $U_{m+1}$. If $g : U \to \mathbb{R}$ is a function of class $C^\infty$ and $(\varepsilon_m)_{m \in \mathbb{N}_*}$ is a sequence of positive real numbers, there exists an analytic function $G : U \to \mathbb{R}$ such that, for every $m \in \mathbb{N}_*$,*

$$|D_\alpha G(p) - D_\alpha g(p)| < \varepsilon_m \text{ for every } p \in U \smallsetminus U_m \text{ and every } \alpha \in \mathbb{N}_*^n \text{ with } |\alpha| \leq m.$$

The combination of this lemma with the *theory of Partitions of Unity* will allow us to build the type of functional extensions we are interested in. We state, without proof, the list of results we need (proofs can be found, for example, in [55, Chapter 2]).

**Definition 1.12.** Let $U \subset \mathbb{R}^n$ be open and $\{U_i\}_{i \in I}$ be an open cover of $U$. A $C^\infty$ *partition of unity subordinate to* $\{U_i\}_{i \in I}$ is a family $\{\varphi_i\}_{i \in I}$ of functions $\varphi_i : U \to \mathbb{R}$ of class $C^\infty$ verifying:

1. for every $i \in I$, $0 \leq \varphi_i \leq 1$ in $U$ with $\mathrm{supp}(\varphi_i) = \mathrm{Cl}(\{p \in U : \varphi_i(p) \neq 0\})$ being compact;

2. every $p \in U$ possesses an open neighbourhood $V_p$ such that $\mathrm{supp}(\varphi_i) \cap V_p \neq \emptyset$ for finitely many indexes $i$;

3. the functional series $\sum_{i\in I}\varphi_i$ gives a well-defined function which is constant and equals 1;

4. $\mathrm{supp}(\varphi_i)\subset U_i$.

**Theorem 1.13** (Existence of Partition of Unity). *If $U$ is an open set in $\mathbb{R}^n$ and $\{U_i\}_{i\in I}$ is an open cover of $U$, then there exists a $C^\infty$ partition of unity subordinate to $\{U_i\}_{i\in I}$.*

Finally, an elementary topological result.

**Lemma 1.14.** *Every open set $U\subset\mathbb{R}^n$ possesses an open cover $\{U_m\}_{m\in N_*}$, with $U_0=\emptyset$, such that, for every $m\in\mathbb{N}_*$, $\mathrm{Cl}(U_m)$ is a compact subset of $U_{m+1}$.*

Let us now fix a function $F:U\to\mathbb{R}$ analytic on an open set $U\subset\mathbb{R}^n$, consider an open cover $\{U_m\}_{m\in\mathbb{N}_*}$ as in the previous lemma and take a $C^\infty$ partition of unity $\{\varphi_m\}_{m\in N_*}$ subordinate to this cover (hence $\varphi_0$ is constantly zero).

For every $m\in\mathbb{N}_*$, find positive real numbers $K_m, L_m$ such that for every $p\in\mathrm{Cl}(U_m)$, every $\alpha\in\mathbb{N}_*^n$ with $|\alpha|\le m$ and every $j\in\mathbb{N}_*$ the following inequalities hold:

$$\begin{aligned}|D_\alpha F(p)| &\le K_m,\\ |D_\alpha\varphi_j(p)| &\le L_m.\end{aligned}$$

We emphasize that the numbers $L_m$ are well-defined because there are finitely many maps $\varphi_j$ whose support intersect $\mathrm{Cl}(U_m)$ (otherwise we would contradict condition 2 in Definition 1.12). We can assume, without loss of generality, that both $(K_m)_m$ and $(L_m)_m$ are increasing.

Consider now the function $\phi:U\to\mathbb{R}$ given by $\phi(p)=\sum_{m=0}^\infty\phi_m(p)$ with

$$\phi_m=\frac{\varphi_m}{L_mK_m2^{2m}}.$$

This function is well-defined and of class $C^\infty$ on $U$ because $\phi$ can be locally written as a finite sum of $C^\infty$ functions. Also, observe that if $p\in\mathrm{Cl}(U_{m+1})\setminus U_m$, then the functions $\phi_0,\dots,\phi_m$ vanish at $p$, and therefore

$$|D_\alpha\phi(p)|\le\frac{1}{L_{m+1}2^m2^{m+1}}$$

whenever $|\alpha|\le m+1$. In particular,

$$|D_\alpha\phi(p)|\le\frac{1}{L_m2^m2^{m+1}}$$

for any $p\in U\setminus U_m$ and any $|\alpha|\le m$.

On the other hand, observe that $\phi$ is positive in $U$. Hence, the numbers

$$\varepsilon_m = \frac{1}{2}\min\left\{\frac{1}{L_m 2^m 2^{m+1}}, \min\{\phi(p) : p \in \mathrm{Cl}(U_{m+1})\}\right\}$$

are positive as well. Applying Lemma 1.11 to the map $\phi$ and the numbers $\varepsilon_m$, we find an analytic function $\Phi : U \to \mathbb{R}$ verifying, for every $m \in \mathbb{N}_*$, the inequality

$$|D_\alpha \Phi(p) - D_\alpha \phi(p)| < \varepsilon_m \text{ for every } p \in U \setminus U_m \text{ and every } \alpha \in \mathbb{N}_*^n \text{ with } |\alpha| \leq m.$$

This implies two things:

$$|D_\alpha \Phi(p)| \leq \frac{1}{L_m 2^m 2^m}$$

for every $p \in U \setminus U_m$ and $|\alpha| \leq m$, and $\Phi(p) > 0$ for every $p \in \mathrm{Cl}(U_{m+1}) \setminus U_m$. The second one means, indeed, that $\Phi$ is positive in $U$ (recall $U_0 = \emptyset$).

Now it is easy to check that the analytic product function $G = \Phi \cdot F$ satisfies the inequality

$$|D_\alpha G(p)| \leq \frac{1}{2^m}$$

for every $p \in U \setminus U_m$, every $m \in \mathbb{N}_*$ and every $\alpha \in \mathbb{N}_*^n$ with $|\alpha| \leq m$. From here, it is already obvious that after extending $G$ to $\mathbb{R}^n \setminus U$ as the zero function, we get a $C^\infty$ function on the whole $\mathbb{R}^n$.

## 1.3 Flows on metric spaces

A *local flow* on a metric space $(X, d)$ is a continuous map $\Phi : \Lambda \subset \mathbb{R} \times X \to X$ satisfying:

- $\Lambda$ is open in $\mathbb{R} \times X$; moreover, for any $z \in X$ the set of numbers $t$ for which $\Phi(t, z)$ is defined is an open interval $I_z := (a_z, b_z)$, with $-\infty \leq a_z < 0 < b_z \leq \infty$;
- $\Phi(0, z) = z$ for any $z \in X$;
- if $\Phi(t, z) = u$, then $I_u = \{s - t : s \in I_z\}$; moreover, $\Phi(r, u) = \Phi(r, \Phi(t, z)) = \Phi(r+t, z)$ for every $r \in I_u$.

In the particular case $\Lambda = \mathbb{R} \times X$, we call $\Phi$ a *flow* on $X$.

Given a local flow $\Phi$ on $X$ we sometimes refer to it as the 2-uple $(X, \Phi)$. We write $\Phi_z(t) = \Phi_t(z) = \Phi(t, z)$ whenever it makes sense, when observe that if $\Phi$ is a flow, then the map $\Phi_t : X \to X$ is a homeomorphism for every $t$. We call $\varphi_\Phi(z) := \Phi_z(I_z)$ *the orbit of* $\Phi$ *through the point* $z$. Here (as for the subsequent notions) we sometimes omit $\Phi$ in the subindex and write $\varphi(z)$ instead. If $\varphi(z) = \{z\}$ (when $I_z = \mathbb{R}$), then we call $z$ a *singular* *point* of $\Phi$ (by $\mathrm{Sing}(\Phi)$ we will denote the union set of all those points); otherwise the orbit, and its points, are called *regular*. Since orbits foliate the space, that is, distinct orbits are disjoint, no point can be regular and singular at the same time — we also say that the union of all orbits of $\Phi$ is a *foliation* of $S$ and that every orbit is a *leaf* of the foliation. When the orbit $\varphi(z)$ is a circle (equivalently, the map $\Phi_z(t)$ is periodic), it is called *periodic*. If $I \subset I_z$ is a interval, then we call $\Phi_z(I)$ a *semiorbit* of $\varphi(z)$ (by an interval we understand a set of the form $(a,b), (a,b], [a.b)$ or $[a,b]$ for some real numbers $a < b$; the empty set and any singleton subset of the real line will be said to be a *degenerated interval*). In the particular cases $I = [a,b]$ (with $\Phi_z(a) = p$, $\Phi_z(b) = q$), $I = [0, b_z)$ or $I = (a_z, 0]$, we rewrite $\Phi_z(I)$ as $\varphi(p,q)$, $\varphi(z,+)$ or $\varphi(-,z)$, respectively.

Let $z$ be a regular point of $\Phi$ and $I \subset I_z$ such that $\Phi_z|_I$ is an injective map. Then, if $J \subset \mathbb{R}$ is open interval and $\rho : J \to \Phi_z(I)$ a homeomorphism, then $\rho^{-1} \circ \Phi_z|_I$ is either strictly increasing (when we say that $\rho$ *preserves the time direction* of $\Phi_z(I)$) or strictly decreasing (when we say that $\rho$ *reverses the time direction* of $\Phi_z(I)$).

Typically, flows are represented geometrically by drawing a set of the plane representing $X$ and decomposing that set in leaves (the orbits of the flow), indicating with an arrowhead the time direction of each of them; we then say that the picture is the *phase portrait* of the flow.

Given $\Omega \subset X$ let us consider for every $p \in \Omega$ the maximal connected subset $J_p \subset I_p$ containing 0 and such that $\Phi_p(J_p) \subset \Omega$. If $J_p$ is an open interval for every $p$, then the restriction of $\Phi$ to $\Delta_\Omega := \{(t,p) \in \Delta : t \in J_p\}$ gives a local flow on $\Omega$, which we will also called (with an abuse of notation) the *restriction of* $\Phi$ *to* $\Omega$. This is always the case if, for example, $\Omega$ is *invariant* for $\Phi$, that is, if it is the union of some orbits of $\Phi$.

We say that two (local) flows $(X_1, \Phi_1)$ and $(X_2, \Phi_2)$ are *locally topologically equivalent* at the points $p_1 \in X_1$, $p_2 \in X_2$, if there is a homeomorphism $h : U_1 \to U_2$ between open neighbourhoods of $p_1$ and $p_2$, with $h(p_1) = p_2$, carrying semiorbits onto semiorbits and preserving time directions — meaning by the latter that the for every regular point $z \in U_1$ of $\Phi_1$ and every open interval $I$ making $\Phi_{1,z}|_I$ injective and $\Phi_{1,z}(I) \subset U_1$, the map $\rho = h \circ \Phi_{1,z}|_I$ preserves the time direction of $h(\Phi_{1,z}(I))$. When the homeomorphism maps the whole $X_1$ onto $X_2$ (hence carrying orbits onto orbits), then we call it a *topological equivalence* between $(X_1, \Phi_1)$ and $(X_2, \Phi_2)$ and say that the flows $(X_1, \Phi_1)$ and $(X_2, \Phi_2)$ are *topologically equivalent.*

If $X$ is additionally assumed to be locally compact and $X_\infty$ is its one-point compactification (see Theorem 1.2), then there exists a (global) flow $(\Phi_\infty, X_\infty)$, having $\infty$ as a singular point, whose restriction $(X, \Phi_\infty)$ is a flow topologically equivalent to $(X, \Phi)$ (in fact, the orbits are the same) — for the details, see, e.g., [44, Lemma 2.3].

We define the *ω-limit set* of the orbit $\varphi(z)$ (or the point $z$) as the set

$$\omega_\Phi(z) := \{u \in X : \exists t_n \to b_z; \Phi_z(t_n) \to u\}\,.$$

*Remark* 1.15. For every $z$ the equality $\omega_\Phi(z) = \bigcap_{0<t<b_z} \mathrm{Cl}\left(\{\Phi_p(s) : t < s < b_z\}\right)$ holds.

The *α-limit set* $\alpha_\Phi(z)$ is analogously defined (now $t_n \to a_z$). Also, we write $\alpha'_\Phi(z) = \alpha_\Phi(z) \setminus \varphi_\Phi(z)$ and $\omega'_\Phi(z) = \omega_\Phi(z) \setminus \varphi_\Phi(z)$. When these sets coincide, that is, the orbit belongs neither to its $\alpha$-limit set not its $\omega$-limit set, we call it *non-recurrent*. If $\varphi_\Phi(z) \subset \alpha_\Phi(z)$ (respectively $\varphi_\Phi(z) \subset \omega_\Phi(z)$) we say that $\varphi_\Phi(z)$ is an *α-recurrent* orbit (respectively an *ω-recurrent* orbit). Then, an orbit is *recurrent* if it is either $\alpha$-recurrent or $\omega$-recurrent (or both). Obviously all singular and all periodic orbits are recurrent: we call these orbits *trivial recurrent* orbits.

In general, both the $\alpha$-limit and the $\omega$-limit set of an orbit are invariant for $\Phi$. Also, notice that if $X$ is compact, every limit set must be nonempty. In particular, if, given a point $z \in X$, the $\omega$-limit (respectively $\alpha$-limit) set of $\varphi_\Phi(z)$ is empty, then $\omega_{\Phi_\infty} = \{\infty\}$ (respectively $\alpha_{\Phi_\infty} = \{\infty\}$).

When, given a point $z \in X$, the $\omega$-limit set (respectively the $\alpha$-limit set) of the orbit $\varphi_{\Phi_\infty}(z)$ is a singleton, say $\omega_{\Phi_\infty}(z) = \{u\}$ (respectively $\alpha_{\Phi_\infty}(z) = \{u\}$) , then $u$ must be necessary a singular point for $\Phi_\infty$ and $\lim_{t\to b_z} \Phi_z(t) = u$ (respectively $\lim_{t\to a_z} \Phi_z(t) = u$). If $\alpha_{\Phi_\infty}(z) = \omega_{\Phi_\infty}(z) = \{u\}$, we say that $\varphi_\Phi(z)$ (or $\varphi_{\Phi_\infty}(z)$) is an *homoclinic* orbit; if $\{u\} = \alpha_{\Phi_\infty}(z) \neq \omega_{\Phi_\infty}(z) = \{v\}$, $\varphi(z)$ is called *heteroclinic*.

Finally, let $u \in X$ be a singular point for $(X, \Phi)$. If $\{u\}$ is the $\omega$-limit (respectively the $\alpha$-limit) set of all the orbits, the we say that $u$ a *globally attracting singular point* or a *global attractor* (respectively *globally repelling singular point* or a *global repeller*). If $u$ possesses a neighbourhood $U$ such that all the orbits in $U \setminus \{u\}$ are periodic, we say that $u$ (and also $U$) is a *center*.

## 1.4 Flows on surfaces

### 1.4.1 Surfaces: definition and some topological properties

#### Definition

By a *surface* (respectively a *surface with boundary* or a *∂-surface*) $S$ we mean a connected, second countable, Hausdorff space such that every point in $S$ possesses an open neighbourhood homeomorphic to some open connected subset of the euclidean plane $\mathbb{R}^2$ (respectively to some open connected subset of the half-space $\mathbb{H}^2 = \mathbb{R} \times [0, +\infty)$).

In a $\partial$-surface $S$, we call *interior points* to those points which have a neighbourhood homeomorphic to some open connected subset of $\mathbb{R}^2$. On the other hand, the points $p \in S$

for which there are an open neighbourhood $U$ of $p$ in $S$, an open connected subset $V \subset \mathbb{H}^2$ and a homeomorphism $f : U \to V$ with $f(p) \in V \cap (\mathbb{R} \times \{0\})$ are called *combinatorial boundary points*. We denote by $\partial S$ the set of all combinatorial boundary points. Notice that a surface $S$ is just a $\partial$-surface with $\partial S = \emptyset$. Every point in a $\partial$-surface is either an interior point or a boundary point but not both at the same time ([55, Theorem 1.37, p. 26]).

More generally, we can talk about manifolds. An *n-manifold* (respectively a *n-manifold with boundary*) $M$ is a Hausdorff, second countable space such that every point in $M$ has an open neighbourhood homeomorphic to an open connected subset of $\mathbb{R}^n$ (respectively to some open connected subset of the half-space $\mathbb{H}^n = \mathbb{R}^{n-1} \times [0, +\infty)$).

A surface (with boundary) is a connected 2-manifold (with boundary). If $S$ is a $\partial$-surface, then $\partial S$ is a 1-manifold [55, p. 27]; therefore, any of the components of $\partial S$ is either homeomorphic to $\mathbb{R}$ or to $\mathbb{S}^1$ [55, p. 398].

Let $S$ be a surface or a $\partial$-surface. A *coordinate chart* on $S$ is a pair $(U, \varphi)$ where $U \subset S$ is open and $\varphi : U \to \varphi(U) \subset \mathbb{R}^2$ is an homeomorphism with $\varphi(U)$ being open in $\mathbb{R}^2$ (if $S$ is a surface) or in $\mathbb{H}^2$ (if $S$ is a $\partial$-surface). We say that $U$ is a *coordinate domain* and $\varphi$ a *coordinate map*; if we write $\varphi$ in components, $\varphi(p) = (x(p), y(p))$ we say that $(x, y)$ are *local coordinates on* $U$.

Given two coordinate charts on $S$, $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$, such that $U_1 \cap U_2 \neq \emptyset$ we call *transition map from* $\varphi_1$ *to* $\varphi_2$, respectively from $\varphi_2$ to $\varphi_1$, to the homeomorphism $\varphi_2 \circ \varphi_1^{-1} = \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ (respectively $\varphi_1 \circ \varphi_2^{-1} = \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$). We say that $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ are $C^r$ *compatible* (with $0 \le r \le \infty$ or $r = \omega$) if either $U_1 \cap U_2 = \emptyset$ or if the associated transitions maps are of class $C^r$. (A real function defined on an open subset $U \subset \mathbb{H}^2$, $f : U \to \mathbb{R}$, is said of class $C^r$ if for every $p \in U$ there exists an open subset $V_p \subset \mathbb{R}^2$ containing $p$ and a map $g : V_p \to \mathbb{R}$ of class $C^r$ which agrees with $f$ on $V_p \cap \mathbb{H}^2$.) We define an *atlas for* $S$ to be a collection of coordinate charts whose domains cover $S$. When any two charts of an atlas are $C^r$ compatible we call it a $C^r$ atlas (or a $C^r$ *structure*) and we say that $S$ is a $C^r$ surface (or a $C^r$ $\partial$-surface if $S$ has a nonempty combinatorial boundary). Two atlas of a surface $S$ are $C^r$ compatible if the union of both atlas is itself a $C^r$ atlas.

**Example 1.16.** The euclidean plane $\mathbb{R}^2$ has a trivial structure of $C^\omega$ surface with the identity map defining a chart for it. We can also easily endow $\mathbb{S}^2$ with an analytic structure using as charts the *stereographic projections* $\pi_N : \mathbb{S}^2 \setminus \{p_N\} \to \mathbb{R}^2$ and $\pi_S : \mathbb{S}^2 \setminus \{p_S\} \to \mathbb{R}^2$ defined, respectively, by $\pi_N(x, y, z) = (x/(1-z), y/(1-z))$ and $\pi_S(x, y, z) = (x/(1+z), -y/(1+z))$ (here $p_N = (0, 0, 1)$ and $p_S = (0, 0, -1)$ are the *north and south poles*).

Let $S$ and $M$ be two $C^r$ surfaces (with or without boundary). A map $F : S \to M$ is said to be of class $C^r$, or a $C^r$ *map*, (for $0 \le r \le \infty$ or $r = \omega$) if for every point $p \in S$, there are coordinate charts $(U, \varphi)$ of $S$ and $(V, \psi)$ of $M$ with $p \in U$, $F(p) \in V$

and $F(U) \subset V$such that $\psi \circ F \circ \varphi^{-1}$ is a map of class $C^r$ in the real euclidean standard sense (as a map from $\varphi(U) \subset \mathbb{R}^2$ to $\psi(V) \subset \mathbb{R}^2$) — again, a map whose domain is a subset of $\mathbb{H}^2$ is understood to be of class $C^r$ if it admits an extension to a $C^r$ map in an open neighbourhood (in $\mathbb{R}^2$) of each point, and a map whose codomain is a subset of $\mathbb{H}^2$ is of class $C^r$ if it is of class $C^r$ as a map into $\mathbb{R}^2$. A bijection map $F : S \to M$ is a $C^r$ *diffeomorphism* (for $0 \leq r \leq \infty$ or $r = \omega$) if both $F$ and $F^{-1}$ are maps of class $C^r$. A map $F : S \to M$ is a local $C^r$ diffeomorphism if every point $p \in S$ has a neighbourhood $U$ such that $F(U)$ is open on $M$ and the restriction of $F$ to $U$ on domain and to $F(U)$ on codomain is a $C^r$ diffeomorphism.

*Remark* 1.17. Any surface (compact or not, with or without boundary) has, up to $C^r$ diffeomorphism (with $r = \infty$ or $r = \omega$), a unique $C^r$ structure and given any two surfaces (or any two $\partial$-surfaces) they are homeomorphic if and only if they are $C^r$ diffeomorphic (see [43, p. 685] and [44, p. 112] for some references). Therefore, whenever we deal with a surface or with a $\partial$-surface we can always consider it equipped with a compatible analytic structure.

*Remark* 1.18. Every surface (with or without boundary) is metrizable. Indeed, let $S$ be a $\partial$-surface. Since $S$ is Hausdorff and locally homeomorphic to an open connected subset of $\mathbb{R}^2$ or of $\mathbb{H}^2$ it follows that it is also regular [62, Lemma 3.1]. Furthermore, because $S$ is second countable, $S$ clearly possesses a basis which is a countable union of locally finite families of subsets and Theorem 1.1 gives the metrizability of $S$.

### Quotient surfaces

Let $(G, +)$ be a group with additive notation for its operation $+$ (we will denote by $0_G$ its identity element) and $S$ be a set. An application $A : G \times S \to S$ is said to be an *action of $G$ on $S$* if for every $g, h \in G$ and every $p \in S$ we have $A(g, A(h, p)) = A(g + h, p)$ and $A(0_G, p) = p$.

Associated with an action we can consider an equivalence relation on $S$ relating any two elements $p, q \in S$ if $A(g, p) = q$ for some $g \in G$. It is standard to denote the quotient under this equivalence relation as $S/A$.

Let us now suppose that $S$ is a $C^\omega$ surface and Let $A : G \times S \to S$ be an action of $G$ on $S$. We say that $A$ is *smooth* (respectively analytic) if for every $g \in G$, the application $A(g, \cdot) : S \to S$ is of class $C^\infty$ (respectively of class $C^\omega$). We say the action $A$ is *free* (or that $G$ *acts on $S$ freely*) if for every $p \in S$, $A(g, p) = p$ if, and only if, $g = 0_G$. Finally the action is called a *proper action* if the map $B : G \times S \to S \times S$ given by $(g, p) \mapsto (A(g, p), p)$ is a proper map; that is, if $K$ is a compact subset of $S \times S$, then $B^{-1}(K)$ is also compact (where in $G$ we are considering the discrete topology).

For a given action with an explicit formula, it is easy in general to guarantee that the action is smooth and free; however, proving that the action is proper can be difficult or at least not an evident task. An interesting characterization of proper actions for some special cases (enough for our purposes) is the following. If $A : G \times S \to S$ is a smooth and free action of a countable (finite or infinite) group $G$ (equipped with the discrete topology) over a surface $S$, then the action is proper if, and only if, for any given sequences $(p_n)_n$ in $S$ and $(g_n)_n$ in $G$ such that $(p_n)_n$ and $(A(g_n, p_n))_n$ are convergent (in $S$), then $(g_n)_n$ possesses a constant subsequence (in $G$) — see [55, Lemma 21.5, p. 543].

**Theorem 1.19** (Quotient Surface Theorem)**.** *Suppose $G$ is a countable (finite or infinite) group acting smoothly (respectively analytically), freely and properly on a $C^\omega$ surface $S$. Then the quotient $S/G$ is a $C^\infty$ (respectively $C^\omega$) surface for which the quotient map $\pi : S \to S/G$ is a local $C^\infty$ (respectively $C^\omega$) diffeomorphism.*

*Proof.* See [55, p. 549, Theorem 21.13] (this reference shows the proof for the case $r = \infty$ but the same proof works word by word for the analytic case). $\square$

**Example 1.20** (Torus)**.** It is an easy exercise to check that the application $A : \mathbb{Z}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $A((l, k), (x, y)) = (x + k, y + l)$ is a free analytic proper action of the group $\mathbb{Z}^2$ (with the sum and with $(0, 0)$ as identity element) on $\mathbb{R}^2$. The analytic surface given by the quotient $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ is known as the *torus*. When using the torus along this dissertation, we will frequently represent it geometrically. To do so, we will draw a unit square $[0, 1] \times [0, 1]$ where the points $(x, 0)$ and $(x, 1)$ (respectively $(0, y)$ and $(1, y)$) are identified for every $0 \leq x \leq 1$ (for every $0 \leq y \leq 1$).

**Example 1.21** (Projective plane)**.** Let us now consider the group $G = \{-1, 1\}$ (with the multiplication as operation and with 1 as identity element). The real *projective plane* can be defined as the quotient $\mathbb{P}^2 := \mathbb{S}^2/\mathbb{Z}_2$ associated with the action $A : \mathbb{Z}_2 \times \mathbb{S}^2 \to \mathbb{S}^2$ given by $G(1, p) = p$ and $G(-1, p) = -p$ for every $p \in \mathbb{S}^2$. Again, it is trivial to show that $A$ is analytic, free and proper and, consequently, $\mathbb{P}^2$ is an analytic surface for which the quotient map $\pi : \mathbb{S}^2 \to \mathbb{P}^2$ associated with the quotient $\mathbb{S}^2/\mathbb{Z}_2$ is a local analytic diffeomorphism.

### Orientation of a surface: a topological definition

We recall that any $\partial$-surface homeomorphic to $\mathbb{S}^1 \times [-1, 1]$ (respectively to $\mathbb{P}^2 \setminus U$ where $U$ is the interior of a disk $D \subset \mathbb{P}^2$) is said to be a *closed annulus* (respectively a *Möbius band*).

We say that a circle $\alpha$ in a surface $S$ is *nonorientable* (respectively *orientable*) if it has a neighbourhood $A \subset S$ with $\mathrm{Cl}(A)$ being a closed annulus (respectively a Möbius band). It is a well-known fact that a circle in a surface is either orientable or nonorientable: we present an elementary proof of this fact based on the classification of surfaces below, see Corollary 1.30.

Given a surface $S$, we say that $S$ is orientable if it does not contain any nonorientable circle; otherwise, $S$ is called nonorientable. A surface with boundary $S$ is said orientable (respectively nonorientable) if the surface $S \setminus \partial S$ is orientable (respectively nonorientable).

**Example 1.22.** It is direct to check that all the circles on the plane, the sphere or the torus are orientable. On the other hand, the projective plane is a nonorientable surface. Indeed, with the notation of Example 1.21, then $\pi(\{(x,y,z) \in \mathbb{S}^2 : z = 0\})$ is clearly a nonorientable circle on $\mathbb{P}^2$.

*Remark* 1.23. Notice that it is a direct consequence of the definition that any two nonorientable circles in $\mathbb{P}^2$ must meet.

### Classification of surfaces

We collect in this section some classification theorems for surfaces. In order to write this section we have consulted mainly the following references: [60, Chapter 1], [33], [71] and [69].

**Definition 1.24.** Given a surface $S$, we say that a compact subset $T \subset S$ is a triangle of $S$ if there exists an homeomorphism $\phi : T' \to T$ with $T'$ being a standard euclidean triangle of straight edges in $\mathbb{R}^2$ (i.e. a compact subset of $\mathbb{R}^2$ bounded by three distinct straight lines). The images, under the homeomorphism $\phi$, of the vertexes (respectively the edges) of the triangle $T'$ are called the *vertexes* (respectively the *edges*) of $T$. A *triangulation* of $S$ is a finite cover of $S$ by triangles of $S$, $\mathcal{T}$, such that any two different triangles are either disjoint, meet in exactly one vertex or meet in a whole edge (c.f. [60, p. 16]).

It is a well-known result in the field of Topology, although it is not easy to prove, that every compact surface has a triangulation (see [80, Theorem 4.1]). The notion of triangulation can be extended to noncompact surfaces (c.f. [60, p. 47]): a extended triangulation for a (noncompact) surface $S$ is a (possibly infinite) cover of $S$ by triangles of $S$, $\mathcal{T}$, such that any two different triangles are either disjoint, meet in exactly one vertex or meet in a whole edge and, furthermore, each point has a neighbourhood that meets only finitely many triangles. With this extended definition it can be proved that every surface (compact or not) admits a triangulation (see [60, Chapter 1, Section 13]).

**Definition 1.25** (Euler characteristic)**.** Let $S$ be a compact surface and $\mathcal{T}$ be a triangulation of $S$. Call $\chi_2$ the number of triangles in $\mathcal{T}$ and $\chi_0$ and $\chi_1$ the number of vertexes and edges of these triangles, respectively. We define the *Euler characteristic of $S$* as

$$\chi(S) = \chi_0 - \chi_1 + \chi_2. \tag{1.2}$$

The formula (1.2) does not depend on the triangulation ([50, Theorem 5.13, p. 105]).

**Definition 1.26** (Genus)**.** Let $S$ be a compact surface. The *genus of $S$* is defined as an integer $g(S)$ according to the following rule:

$$g(S) = \begin{cases} \frac{1}{2}(2 - \chi(S)) & \text{if } S \text{ is orientable,} \\ 2 - \chi(S) & \text{if } S \text{ is nonorientable.} \end{cases} \tag{1.3}$$

Clearly, the Euler characteristic, and therefore the genus, of a surface stays constant under homeomorphisms.

**Example 1.27.** It is a trivial exercise to check that $g(\mathbb{S}^2) = 0$ and $g(\mathbb{T}^2) = g(\mathbb{P}^2) = 1$.

Given two compact surfaces, $S_1$ and $S_2$, we define their *connected sum*, denoted by $S_1 \# S_2$, as follows. We first choose disks $D_1 \subset S_1$ and $D_2 \subset S_2$ and a homeomorphism $f$ from $\partial D_1$ to $\partial D_2$. Call $S_1' = S_1 \smallsetminus \mathrm{Int}(D_1)$ and $S_2' = S_2 \smallsetminus \mathrm{Int}(D_2)$. Finally $S_1 \# S_2$ is defined as the quotient space of $S_1' \cup S_2'$ by identifying any point with itself and the points $x \in \partial D_1$ with $f(x) \in \partial D_2$. $S_1 \# S_2$ is a compact surface with the quotient topology (see [55, Attaching Smooth Manifolds Along Their Boundaries]). Moreover, it can also be proved that the topology of $S_1 \# S_2$ does not depend either on the disks $D_1$ and $D_2$ or on the homeomorphism $f$ used in its definition (that is, if one chooses two different pairs of disks in the given surfaces and builds two quotients as above, then both resultant surfaces are homeomorphic). A formula to compute the Euler characteristic of the connected sum of $S_1$ and $S_2$ is ([60, Theorem 8.1]):

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

We already have all the ingredients we need to state the classification of compact surfaces.

**Theorem 1.28** (Classification of compact surfaces)**.** *Two compact surfaces surfaces are homeomorphic if and only if they have the same Euler characteristic and the same genus (equivalently, if they have the same character of orientability, and either they have the same Euler characteristic or they have the same genus).*

*With more detail:*

1. *If $S$ is a compact orientable surface of genus $g$ then $S$ is homeomorphic to a connected sum of $g$ tori ($g = 0, 1, 2 \ldots$);*

2. *If $S$ is a compact nonorientable surface of genus $g$ then $S$ is homeomorphic to a connected sum of $g$ projective planes ($g = 1, 2 \ldots$);.*

An orientable compact surface of genus $g$ is a sphere with $g$ ‘handles’; a nonorientable compact surface of genus $g$ has $g$ *cross-caps* (i.e., if $S$ is written as a connected sum of projective planes, then that sum consists of exactly $g$ of these projective planes). Equivalently, for any $2k < g$, $S$ has $k$ handles and $g - 2k$ cross-caps. In other words, on a nonorientable surface a pair of cross-caps is equivalent to a handle (as long as at least one cross-cap is left).

In what follows, for every non-negative (respectively positive) integer $g$, $M_g$ (respectively $N_g$) denotes the only, up to homeomorphisms, orientable (respectively nonorientable) compact surface of genus $g$. Recall that the Euler characteristic, $\chi(\cdot)$, can be computed as $\chi(M_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$.

With this notation, $M_0$ (respectively $M_1$, $N_1$) equals, up to homeomorphisms, the sphere $\mathbb{S}^2$ (respectively, the torus $\mathbb{T}^2$, the projective plane $\mathbb{P}^2$). We also represent by $\mathbb{B}^2$ the surface $N_2$, the *Klein bottle*.

If we select a finite number of disjoint disks in a compact surface and remove their interiors, then we obtain a surface with boundary. The number of boundary components is equal to the number of subtracted disks. Conversely, if $S$ is a compact surface with boundary and the boundary has $k$ components ($k$ circles), it is clear that, after gluing $k$ disks to these circles we obtain a standard compact surface (see [55, Attaching Smooth Manifolds Along Their Boundaries]). Summarizing, the collection of compact surfaces with boundary coincide with the collection of compact surfaces from which we subtract a finite number of disks has been removed.

If $S$ is a compact surface with boundary which has exactly $k$ boundary components and $\bar{S}$ is the compact surface we obtain $S$ after gluing $k$ disks to these boundary components, then we define the Euler characteristic of $S$ as the integer $\chi(S) = \chi(\bar{S}) - k$. The genus of $S$ is, however, defined as the genus of $\bar{S}$: with this convention, an orientable (respectively nonorientable) surface with boundary still has as many handles (respectively cross-caps) as its genus indicates.

**Theorem 1.29** (Classification of compact surfaces with boundary)**.** *Let $S_1$ and $S_2$ be compact surfaces with boundary, and assume that their boundaries have the same number of components. Then $S_1$ and $S_2$ are homeomorphic if and only if the obtained compact surfaces after gluing a disk to each boundary component are homeomorphic. In other words, two compact surfaces with boundary are homeomorphic if and only if they have the same number of boundary components, the same Euler characteristic, and both of then are either orientable or nonorientable.*

As a consequence of this classification, we are now ready to prove that any circle in a surface is either orientable or nonorientable.

**Corollary 1.30.** *Any circle in a surface possesses a neighbourhood which is homeomorphic either to a compact annulus or to a Möbius band.*

*Proof.* Let $\alpha$ be a circle in a surface $S$. The compactness of $\alpha$ and the properties of $S$ and $\alpha$ as manifolds allow us to consider a finite cover for $\alpha$ by open coordinate neighbourhoods $\{(U_i, \varphi_i)\}_{i=1}^m$, for some $m \geq 2$, such that for every $i$, $\alpha$ meets every $U_i$ is a unique open arc and $\varphi_i(U_i) = B(0,1)$.

Associated with any $i \in \{1, 2, \ldots, m\}$, we call $i_-$ and $i_+$ the only two integers in $\{1, 2, \ldots, m\}$ such that $i_- + 1 = i = i_+ - 1 \pmod{m}$. Without loss of generality, taking a smaller coordinate neighbourhood if necessary, we may assume that for every $i, j \in \{1, 2, \ldots, m\}$:

- $\varphi_i$ extends to a homeomorphism $\bar{\varphi}_i : \mathrm{Cl}(U_i) \to \mathrm{Cl}(B(0,1))$;
- $V_{i,j} = \mathrm{Cl}(U_i) \cap \mathrm{Cl}(U_j)$ is nonempty if and only if $j \in \{i_-, i_+\}$; moreover, in such a case, $V_{i,j}$ is a disk and $V_{i,j} \cap \alpha$ is also nonempty (and consists of an arc).

We notice that both $D = \cup_{1 \leq i \leq m-1} \mathrm{Cl}(U_i)$ and $\mathrm{Cl}(U_m)$ are disks whose intersection consists in two connected components: two disks. Hence, $A = D \cup \mathrm{Cl}(U_m)$ is a compact $\partial$-surface of Euler characteristic $\chi(A) = \chi(D) + \chi(\mathrm{Cl}(U_m)) - \chi(D \cap \mathrm{Cl}(U_m)) = 1 + 1 - 2 = 0$.

Moreover, it is easy to see that $\partial A$ has either two or one components (circles in any case). After attaching disks to $A$ along the components of its combinatorial boundary we get a compact surface. In the former (respectively latter) case, we get a compact surface $M$ with Euler characteristic given by $\chi(M) = 2$ (respectively by $\chi(M) = 1$) so $M$ is homeomorphic to $\mathbb{S}^2$ (respectively to $\mathbb{P}^2$). $\square$

In order to introduce an analogous classification for noncompact surfaces we first need some notation.

Let $S$ be a $\partial$-surface. A subset $T \subset S$ which, with the induced topology of $S$, has itself structure of $\partial$-surface is said to be a *$\partial$-subsurface* of $S$ (or simply a *subsurface* if $\partial T = \emptyset$).

A $\partial$-surface $S$ is said to be *planar* if every compact $\partial$-subsurface $T \subset S$ has genus zero. Every planar surface is necessarily orientable (every nonorientable surface has genus greater than zero). A $\partial$-surface $S$ is said to be of *finite genus* if there exists a compact $\partial$-subsurface $T \subset S$ such that $S \setminus T$ is planar; in such a case we define the genus of $S$ to be the genus of $T$.

A $\partial$-surface $S$ is said of *infinite genus* (respectively *infinitely nonorientable*) if there is no compact subset $A$ of $S$ such that $S \setminus A$ is of genus zero (respectively orientable). Clearly an infinitely nonorientable surface is also of infinite genus.

A subset of a noncompact $\partial$-surface $S$ is said *bounded* (respectively *unbounded*) if its closure (in $S$) is compact (respectively noncompact).

**Definition 1.31** (Generalized boundary components)**.** Let $S$ be a noncompact surface. A *generalized boundary component of* $S$ is a nested sequence $P_1 \supset P_2 \supset \cdots P_n \supset \cdots$ of unbounded regions in $S$ such that:

1. the boundary of $P_n$ in $S$ is compact for all $n$;
2. for any bounded subset $A$ of $S$, $P_n \cap A = \emptyset$ for $n$ sufficiently large.

In what follows, $S$ is supposed to be a fixed noncompact surface. Two generalized boundary components $P_1 \supset P_2 \supset \cdots$ and $P'_1 \supset P'_2 \supset \cdots$ of $S$ are *equivalent* if, for any $n \in \mathbb{N}$, there are $n_1, n_2 \in \mathbb{N}$ such that $P_{n_1} \subset P'_n$ and $P'_{n_2} \subset P_n$. This relation between generalized boundary components induces an equivalence relation. If $p = \{P_1 \supset P_2 \supset \cdots\}$ is a generalized boundary component of $S$, we will denote by $[p]$ its equivalence class. Any of these equivalence classes is called an *end of a generalized boundary component of $S$*.

Given any $U \subset S$ whose boundary in $S$ is compact, we define $U^*$ to be the set of all ends $[p]$, represented by some $p = \{P_1 \supset P_2 \supset \cdots\}$, such that $P_n \subset U$ for all sufficiently large $n$. It is clear that this is a good definition: it does not depend on the representative of $[p]$. We collect all these sets $U^*$ in a collection $\mathcal{B}(S)$; it is an easy exercise to see that $\mathcal{B}(S)$ is the basis for some topology. The *ideal boundary $B(S)$ of $S$* is the set of all equivalence classes of generalized boundary components of $S$ equipped with the topology having $\mathcal{B}(S)$ as basis.

Let $p = \{P_1 \supset P_2 \supset \cdots\}$ be a generalized boundary component of $S$. We say that $[p]$ is planar (respectively orientable) if the sets $P_n$ are planar (respectively orientable) for all sufficiently large $n$. It is obvious that these notion does not depend on the representative member of $[p]$.

Two important subsets of $B(S)$ are:

$$\begin{aligned} B'(S) &= \{[p] \in B(S) : [p] \text{ is not planar}\}, \\ B''(S) &= \{[p] \in B(S) : [p] \text{ is nonorientable}\}. \end{aligned}$$

It can be proved that $B''(S) \subset B'(S) \subset B(S)$ is a nested sequence of totally disconnected, separable and compact (metric) spaces.

Suppose now that $S$ is neither orientable nor infinitely nonorientable. Then every sufficiently large compact subsurface of $S$ has genus of the same parity: if that parity is even (respectively odd), then we say that $S$ if of even (respectively odd) orientability type. So among the class of noncompact surfaces we distinguish four classes: the orientability, the infinitely nonorientability, the odd orientability and the even orientability classes.

**Theorem 1.32** (Kerékjártó Theorem)**.** *Ñet $S_1$ and $S_2$ be two surfaces of the same genus and orientability class. Then $S_1$ and $S_2$ are homeomorphic if, and only if, there is an homeomorphism $h : B(S_1) \to B(S_2)$ such that $h(B'(S_1)) = B'(S_2)$ and $h(B''(S_1)) = B''(S_2)$.*

I. Richards, in [71], proved that any triple nested sequence of compact, separable, totally disconnected spaces $Z \subset Y \subset X$ occurs as the ideal boundary of some surface: he explicitly built such a surface as a sphere, punctured by a certain set of points and open disks, with specified boundary identifications for the disks.

**Theorem 1.33** (Richards)**.** *Every surface is homeomorphic to a surface obtained from the sphere $\mathbb{S}^2$ by first removing a compact totally disconnected set $X$ from $\mathbb{S}^2$, then removing the interior of a finite or infinite sequence $(D_i)_i$ of disjoint disks in $\mathbb{S}^2 \setminus X$, and finally suitably identifying the boundaries of these disks in pairs (it may be necessary to identify the boundary of one disk with itself to produce a cross-cap). Moreover, when the sequence $(D_i)_i$ is infinite, we have that for any open subset $U \subset \mathbb{S}^2$ containing $X$, all but a finite number of the $D_i$ are contained in $U$.*

The proof of this theorem in [71] is such that the genus of the built surface only depends on the identifications between the boundaries of the subtracted disks of the sphere. This fact allows us to give a more descriptive result for the case of finite genus surfaces.

**Corollary 1.34.** *Let $S$ be an orientable (respectively nonorientable) surface of finite genus $g$. Then, for any compact orientable (respectively nonorientable) surface of genus $g$, $M$, there exists a totally disconnected subset $K \subset M$ such that $M \setminus K$ is homeomorphic to $S$. Moreover, if $L$ is a totally disconnected subset of $M$ which is homeomorphic to $K$, then $M \setminus L$ is also homeomorphic to $S$.*

*Proof.* This is a direct consequence of the previous theorem and the Kerékjártó Theorem after noticing that in a surface of finite genus all equivalent classes of generalized boundary components are planar. □

*Remark* 1.35. This corollary shows that every noncompact surface of finite genus possesses a compactification which is itself a surface. In general, given a noncompact surface $S$, an embedding $h : S \to M$ of $S$ into a topological space $M$ is said to be a *compactification of $S$* if $M$ is compact and $h(S)$ is an open and dense subset of $M$. Given a noncompact orientable (respectively nonorientable) surface $S$ of finite genus $g$, Corollary 1.34 says that there exists an embedding $h : S \to M$ with $M = M_g$ (respectively $M = N_g$) such that $K = M \setminus h(S)$ is totally disconnected. So, in particular, $h$ is a compactification of $S$. The set $M$ is also locally connected and Hausdorff (it is a surface) and $K$ is *nonseparating* on $M$ (i.e., for any region $U \subset M$, the set $U \setminus K$ is also connected). This additional property implies that the compactification $h$ is unique in the following sense. If $M'$ is any other compact Hausdorff and locally connected space for which there exists an embedding $h' : S \to M'$ making $K' = M' \setminus h'(S)$ being totally disconnected and nonseparating on $M'$, then there exists a homeomorphism $F : M \to M'$ such that $(h')^{-1} \circ F \circ h$ is the identity map on $S$.

### 1.4.2 Some differentiable properties of surfaces

#### Tangent spaces and vector fields

Let us fix for the whole section a $C^r$ surface (with or without boundary), $S$, for some $1 \leq r \leq \infty$ or $r = \omega$. We will denote by $C^r(S)$ the set of all functions $f : S \to \mathbb{R}$ of class $C^r$.

Given any $p \in S$, we will say that a $\mathbb{R}$-linear map $v : C^r(S) \to \mathbb{R}$ is a *derivation at* $p$ if it satisfies the equation $v(fg) = f(p)v(g) + g(p)v(f)$ for every $f, g \in C^r(S)$. The set of all derivations at $p$, denoted by $T_pS$, is called the *tangent space of* $S$ *at* $p$, the elements of $T_pS$ being the *tangent vectors* of $S$ at $p$, and the union set $TS = \cup_p T_pS$ is called the *tangent bundle* of $S$ . It can be proved that, for every $p$, $T_pS$ is a $\mathbb{R}$-vector space of dimension 2 and that $TS$ has structure of $C^r$ surface (with boundary if $\partial S \neq \emptyset$).

If $(U, \varphi)$ is a coordinate chart on $S$ with $p \in S$ and $x, y : U \to \mathbb{R}$ are the associated local coordinates, we consider two derivations $\frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\big|_p \in T_pS$ given by the formulas

$$
\begin{aligned}
\left.\frac{\partial}{\partial x}\right|_p (f) &:= \frac{\partial(f \circ \varphi^{-1})}{\partial x}(\varphi(p)), \\
\left.\frac{\partial}{\partial y}\right|_p (f) &:= \frac{\partial(f \circ \varphi^{-1})}{\partial y}(\varphi(p)),
\end{aligned}
$$

for every $f \in C^r(S)$. It can be proved that $\left\{ \frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\big|_p \right\}$ is a base for $T_pS$; given any derivation at $p$, $v$, it can be expressed as a linear combination as

$$
v = v(x) \left.\frac{\partial}{\partial x}\right|_p + v(y) \left.\frac{\partial}{\partial y}\right|_p .
$$

A $C^r$ *vector field* on $S$ is a map $X : S \to TS$ with $X(p) \in T_pS$ for every $p \in S$ and such that for every coordinate chart $(U, \varphi = (x, y))$, the functions $q \mapsto X_q(x)$ and $q \mapsto X_q(y)$ are of class $C^r$ on $U$, where, for every $q \in U$, $X_q(x)$ and $X_q(y)$ are the only two real numbers verifying

$$
X(q) = X_q(x) \left.\frac{\partial}{\partial x}\right|_q + X_q(y) \left.\frac{\partial}{\partial y}\right|_q ,
$$

that is, $X_q(x) := (X(q))(x)$ and $X_q(y) := (X(q))(y)$.

Let us now fix a $C^r$ vector field on $S$, $X$, and take a curve $\gamma : I \to S$ of class $C^r$ defined in an open interval $I \subset \mathbb{R}$. Given any $s \in I$, we define the *velocity of* $\gamma$ *at* $s$, denoted as $\gamma'(s)$, as the derivation at $\gamma(s)$ given by the formula

$$
\gamma'(s)(f) := \left.\frac{d}{dt}(f(\gamma(t)))\right|_{t=s}, \text{ for every } f \in C^r(S).
$$

We then say that $\gamma$ is an *integral curve* of $X$ if for every $s \in I$ we have $\gamma'(s) = X(\gamma(s))$.

*Remark* 1.36. Finding the integral curves associated with a given $C^r$ vector fields is equivalent to solving autonomous systems of differential equations. Indeed, let $X$ be a $C^r$ vector field on $S$ and $\gamma : I \to S$ be a curve of class $C^r$ defined on an open interval $I \subset \mathbb{R}$. Let $s \in I$ and consider a coordinate chart $(U, \varphi = (x, y))$ with $\gamma(s) \in U$, and an $\epsilon_s > 0$ such that $\gamma((s - \epsilon_s, s + \epsilon_s)) \subset U$ and name as $\gamma_1$ and $\gamma_2$ the coordinates of the composition $\varphi \circ \gamma|_{(s-\epsilon,s+\epsilon)}$ (i. e. $\varphi(\gamma(t)) = (\gamma_1(t), \gamma_2(t))$ for every $t \in (s - \epsilon_s, s + \epsilon_s)$). Hence, the condition $\gamma'(s) = X(\gamma(s))$ translates to solving the equation

$$\dot{\gamma_1}(s) \left.\frac{\partial}{\partial x}\right|_{\gamma(s)} + \dot{\gamma_2}(s) \left.\frac{\partial}{\partial y}\right|_{\gamma(s)} = X_{\gamma(s)}(x) \left.\frac{\partial}{\partial x}\right|_{\gamma(s)} + X_{\gamma(s)}(y) \left.\frac{\partial}{\partial y}\right|_{\gamma(s)},$$

or what it is the same, to resolving the system

$$\begin{cases} \dot{\gamma_1}(s) = X_{\gamma(s)}(x), \\ \dot{\gamma_2}(s) = X_{\gamma(s)}(y), \end{cases} \tag{1.4}$$

(where, if $f : I \to \mathbb{R}^m$ is a $C^1$ function on an open interval of $\mathbb{R}$, we use $\dot{f}(s)$ to denote its derivative at $s$).

This last remark is the key to transfer the theory of ordinary differential equations on the plane to general surfaces. For example the existence and uniqueness of solutions for $C^r$ planar autonomous differential equations is translated into the following result.

**Theorem 1.37.** *Let $X$ be a $C^r$ vector field on a $C^r$ surface (for some $1 \leq r \leq \infty$ or $r = \omega$). For every point $p \in M$, there exists a unique maximal $C^r$ curve $\gamma_p : I_p \to S$, defined in an open interval $I_p \subset \mathbb{R}$ containing $0$, which is an integral curve of $X$ starting at $p$ (that is, a integral curve of $X$ with $\gamma_p(0) = p$ and such that, if $\bar{\gamma} : \bar{I} \to S$ is also an integral curve of $X$ with $0 \in \bar{I}$ and $\bar{\gamma}(0) = p$, then $\bar{I} \subset I$ and $\bar{\gamma}(t) = \gamma(t)$ for every $t \in \bar{I}$).*

With the notation of this last theorem, we will say that $\gamma_p$ is *the (maximal) integral curve of $X$ starting at $p$*.

The details for the proof of this theorem (for the cases $1 \leq r \leq \infty$) can be found, for example, in [55, Chapter 8]. But the proof also work for the analytic case: we just have to highlight that the existence of uniqueness results for $C^r$ planar autonomous differential equation also hold for $r = \omega$ (for example, see [56, pp. 43–45] for a discussion on the details needed to adapt the $C^\infty$ proof to this case; in [24] we presented a self-included proof of this fact).

### Differentials and embeddings

Let $S$ and $M$ be two $C^r$ surfaces with or without boundary (for some $1 \leq r \leq \infty$ or $r = \omega$) and $F : S \to M$ be a map of class $C^r$. Given any $p \in S$ and any derivation at $p$, $v$, it is direct to check that the map $d_pF(v) : C^r(M) \to \mathbb{R}$ given by $d_pF(v)(f) := v(f \circ F)$ for every $f \in C^r(M)$ is a derivation at $F(p)$. We then define the *differential of $F$ at $p$* as the $\mathbb{R}$-linear map $d_pF : T_pS \to T_{F(p)}M$ associating, to any $v$, the derivation $d_pF(v)$. The map $dF : TS \to TM$ given by $dF(v) := d_pF(v)$ is then said to be the *differential of $F$*.

The map $F$ is said to be a $C^r$ *embedding* if it is a topological embedding (that is, an injective continuous map which becomes a homeomorphism after restricting its codomain to its range $F(S)$) and, for every $p$, $d_pF$ is injective.

### Orientation of a surface: a differentiable definition

We introduce here a definition of orientability for surfaces (or $\partial$-surfaces) based on the notion of vector fields. This new definition is equivalent to that one introduce in Section 1.4.1.

We start by recalling the analogous concept of orientability for real vector spaces. Suppose that $V$ is vector space over $\mathbb{R}$ of finite dimension $n \geq 1$ and let $\mathcal{B}_1 = (v_1, \dots, v_n)$ and $\mathcal{B}_2 = (w_1, \dots, w_n)$ be two (ordered) bases for $V$. Given any base $\mathcal{B}$ of $V$ and any $v \in V$ we will denote by $[v]_{\mathcal{B}}$ the vector in $\mathbb{R}^n$ giving the coordinates of $v$ respect the base $\mathcal{B}$. Fix any base of $V$, $\mathcal{B}$, and let $T : V \to V$ be the unique linear automorphism satisfying $T(v_i) = w_i$ for all $1 \leq i \leq n$ and call $\mathcal{M}_T$ its matrix representation with respect a base $\mathcal{B}$ (i.e. $\mathcal{M}_T$ is the unique square matrix of order $n$ satisfying that for every $v \in V$, $[T(v)]^t_{\mathcal{B}} = \mathcal{M}_T\, [v]^t_{\mathcal{B}}$ where $[v]^t_{\mathcal{B}}$ is simply the vector $[v]_{\mathcal{B}}$ written in column notation). We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *consistently oriented* if $\mathcal{M}_T$ has positive determinant. Being consistently oriented defines in the family of bases for $V$ an equivalence relation; we defined an *orientation for $V$* as an equivalence class of that relation. A real vector space $V$ together with a choice of orientation is called an *oriented vector space.*

Given a surface (or a $\partial$-surface) $S$, we define a *point-wise orientation on $S$*, $\mathcal{O}$, to be a choice of orientation of each of its tangent spaces (i.e. $\mathcal{O}_p$ is an orientation of the 2-dimensional real vector space $T_pS$ for each $p \in S$). We say that a point-wise orientation $\mathcal{O}$ is *continuous* if for every point $p$ of $S$ there exist an open neighbourhood $U \subset S$ of $p$ and two $C^1$ vector fields on $U$, $X_1$ and $X_2$ such that $(X_1|_p, X_2|_p)$ is a positively oriented basis for $T_pS$ (that is, $(X_1|_p, X_2|_p)$ belongs to the equivalence class of $\mathcal{O}_p$). An *orientation of $S$* is simply a continuous point-wise orientation; $S$ is said to be orientable if there exists an orientation for it and nonorientable otherwise. An orientable surface with a fixed orientation is called an *oriented surface.*

A coordinate chart $(U, \varphi)$, with $\varphi = (x, y)$, of an oriented surface (or $\partial$-surface) $S$

is said to be positively oriented if for every $p \in U$ the 2-uple of the coordinate tangent vectors $\left( \frac{\partial}{\partial x}\Big|_p , \frac{\partial}{\partial y}\Big|_p \right)$ is positively oriented. An atlas for $S$ is said *consistently oriented* if every two non-disjoint coordinate chars have associated transition maps with positive Jacobian determinant everywhere on its domains. A surface is orientable if, and only if, it possesses a consistently oriented $C^\omega$ atlas ([55, Proposition 15.6, p. 382]) — the result just cited is given and proved for $C^\infty$ regularity but everything is clearly extended in an obvious way to the $C^\omega$ class.

A helpful criterion for establishing the orientability of a particular surface is the following one. A surface (or a $\partial$-surface) $S$ is orientable if, and only if, it admits an ordered 2-upla of linearly independent continuous vector fields $(X, Y)$ with domain the whole surface $S$ (i.e. if $X_p$ and $Y_p$ are linearly independent vectors of $T_pS$ for every $p \in S$) — see [55, Theorem 15.17].

The previous characterization could of course be used as a nonorientability criterion but it is easy to understand that it is almost not practical at all: it can be easy to, for a given surface, find a particular pair of linearly independent continuous vector fields defined on the whole surface but proving that such a pair cannot exists can be a extremely tough task. A standard alternative can be trying to find a nonorientable circle in the surface, that is, to use the definition of orientability given in the previous section.

### Poincaré-Hopf index Theorem

In general, computing the Euler characteristic of a given surface using the definition can be a very tedious task. We present here a very useful alternative for surfaces where a $C^1$ vector field with finitely many singularities is defined: the so-called *Poincaré-Hopf index Theorem*. In order to present a precise statement, we begin by introducing some needed notions, definitions and auxiliary results.

Let $X$ and $\tilde{X}$ be arcwise connected and locally arcwise connected spaces and let $p : \tilde{X} \to X$ be continuous. The pair $(\tilde{X}, p)$ is called a *covering space of* $X$ if for every $x \in X$ there exists an open neighbourhood $U$ of $x$ such that $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto $U$ by $p$. In such a case we also say that $p$ is a covering map between $\tilde{X}$ and $X$ and that $\tilde{X}$ is a covering for $X$.

*Remark* 1.38. With the notation above:

1. $p$ is an open map and a local homeomorphism;

2. Since $X$ is connected, if $p^{-1}(\{x_0\})$ has $k \in \mathbb{N}$ elements for some $x_0 \in X$, then $p^{-1}(\{x\})$ has also $k$ elements for every $x \in X$. In such a case, we say that $(\tilde{X}, p)$ is a $k$-fold covering of $X$;

3. If $S$ is a surface (or more in general a $n$-manifold), then so is $\tilde{S}$. Moreover, if $(\tilde{S}, p)$ is a $k$-fold covering of a surface $S$, then the Euler characteristics of $S$ and $\tilde{S}$ are related by the formula $\chi(\tilde{S}) = k\chi(S)$ (see [50, Theorem 7.19, p. 167]).

**Example 1.39.** Two easy examples:

1. If $X = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, the pair $(\mathbb{R}, p)$, with $p : \mathbb{R} \to \mathbb{S}^1$ given by $p(t) = \exp(2\pi i)$, is a covering space for $X$;
2. Let $X = \mathbb{P}^2$ be the projective plane and. Let us call $\tilde{X} = \mathbb{S}^2$ and take $p : \tilde{X} \to X$ defined by $p(z) = \{z, -z\}$. The pair $(\tilde{X}, p)$ is a covering space of $X$.

**Theorem 1.40** (Double covering of nonorientable surfaces)**.** *Every (compact) nonorientable surface (respectively $\partial$-surface) $S$ has a 2-fold covering space $(\tilde{S}, p)$ with $\tilde{S}$ being a (compact) orientable surface (respectively $\partial$-surface).*

*Proof.* See for example [55, pp. 393–395]. □

The 2-fold covering of the previous theorem is in fact achieved as a $C^\omega$ locally diffeomorphic map and is essentially unique in the sense that if $(\tilde{S}, p)$ and $(S', q)$ are two 2-fold covering spaces as in the previous statement such that $p$ and $q$ are locally diffeomorphisms, then there exists a unique orientation-preserving diffeomorphism $\varphi : S' \to \tilde{S}$ such that $\varphi \circ p = q$ ([55, Proposition 15.42, p. 396]).

**Corollary 1.41.** *If $N_g$ denotes the nonorientable compact surface of genus $g$, then there exists a covering map between $M_{g-1}$ and $N_g$.*

*Proof.* According to Theorem 1.40, $N_g$ has a 2-fold covering space $(\tilde{N}_g, p)$ with $\tilde{N}_g$ being a compact orientable surface. Remark 1.38 says then that $\chi(\tilde{N}_g) = 2\chi(N_g) = 2(2-g) = 2(1-(g-1))$. By the classification theorem for compact surfaces we conclude that $\tilde{N}_g$ is homeomorphic to a connected sum of $g-1$ torus. □

Let $S$ be an oriented compact surface, $X$ be a $C^1$ vector field on $S$ and $p$ be an isolated singular point of $X$. Associated with $p$ we can take a small disk such that $p$ is the only zero of $X$ in $\mathrm{Int}(B_p)$ and $\mathrm{Int}(B_p)$ is a coordinate domain — let $\varphi = (x, y)$ be a coordinate map associated with $\mathrm{Int}(B_p)$. For every $q \in \mathrm{Int}(B_p)$, let $V(p) \in \mathbb{R}^2$ be the coordinates of $X(q)$ with respect to the basis $\left\{ \frac{\partial}{\partial x}\Big|_q, \frac{\partial}{\partial y}\Big|_q \right\}$ of $T_qS$. The *Gauss map on $B_p$* (associated with $X$) is then defined to be the map $g_p : \mathrm{Int}(B_p) \setminus \{p\} \to \mathbb{S}^1$ given by $g_p(q) = \frac{1}{\|V(q)\|} V(q)$. The *index of the point $p$* (with respect to the vector field $X$), denoted as $\mathrm{ind}_X(p)$, is defined as the degree of the map $g_p$. This last number coincides with the winding number of the map $g_p$ when is restricted to $\partial B_p$ (for a formal definition see [21, p. 112]). When the point $p$ is a non-degenerate singularity of $X$ (that is, when the determinant of the Jacobian of $X$

in $p$, computed with respect to any compatible chart, does not vanish), then the index of $p$ can be computed simply as the sign of the product of the eigenvalues of the Jacobian of $X$ in $p$ (see [21, Theorem 14.4.3.]).

**Example 1.42.** The index of an isolated critical point which has a neighbourhood with a finite sectorial decomposition with exactly $h$ hyperbolic sectors, $e$ elliptic sectors and $p$ parabolic sectors is given by the formula $\frac{1}{2}(2 - h + e)$ (see [7, Theorem 4.1, p. 36]) — for the definition of finite sectorial decomposition, see p. 35 below.

**Theorem 1.43** (Poincaré Index Theorem)**.** *Let $S$ be a compact orientable surface of genus $g$ and $X$ be a $C^1$ vector field on $S$. If $X$ possesses only finitely many singular points, say $p_1, \dots, p_n$, then $2 - 2g = \chi(S) = \sum_{i=1}^{n} \mathrm{ind}_X(p_i)$.*

*Proof.* See [21, Theorem 15.2.7]. □

**Corollary 1.44** (Poincaré-Hopf Index Theorem)**.** *If a $C^1$- vector field $X$ on a nonorientable compact surface $S$ of genus $g$ has only finitely many singular points, say $p_1, \dots, p_n$, then the Euler characteristic of $S$ must verify the relation $2 - g = \chi(S) = \sum_{i=1}^{n} \mathrm{ind}_X(p_i)$.*

*Proof.* Let $(\tilde{S}, p)$ be a double covering of $S$. The surface $\tilde{S}$ is a connected compact orientable surface of genus $g - 1$ and $p : \tilde{S} \to S$ is a local diffeomorphism.

The map $p$ allows us to defined a $C^1$ vector field over $\tilde{S}$ in a natural way. Let $q' \in \tilde{S}$, $q = p(q') \in S$ and $U$ and $V$ be open neighbourhoods of $q'$ and $q$ respectively such that by restricting $p$ to $U$ one obtains a diffeomorphism from $U$ to $V$. Let $dp_{q'}$ denote the isomorphism that $p$ generates between the tangent space of $U$ on $q'$, $T_{q'}U$, and of $V$ on $q$, $T_qV$. Then we define $\tilde{X}(q') = (dp_{q'})^{-1}(X(q))$. It is then clear that $\tilde{X}$ is a $C^1$ vector field on $\tilde{S}$ with exactly $2n$ isolated singular points (every $p_i$ generates two singular points for $\tilde{V}$, say $\tilde{p}_i$ and $\tilde{p}_{n+i}$). Moreover, for every $1 \leq i \leq n$, $\mathrm{ind}_{\tilde{X}}(\tilde{p}_i) = \mathrm{ind}_{\tilde{X}}(\tilde{p}_{n+i}) = \mathrm{ind}_X(p_i)$. Applying Theorem 1.43 to $\tilde{X}$, we have $2\chi(S) = \chi(\tilde{S}) = \sum_{i=1}^{2n} \mathrm{ind}_{\tilde{X}}(\tilde{p}_i) = 2\sum_{i=1}^{n} \mathrm{ind}_X(p_i)$ as was required. □

#### Some useful deeper results on analyticity on open subsets of a surface

Let $S$ be an analytic surface and fix a region $O \subset S$. We say that a set $A \subset O$ is *analytic* (in $O$) if it is the set of zeros of some analytic map $f : O \to \mathbb{R}$. Later in the dissertation we will consider unions of analytic sets in open subsets of the sphere. In general, the union of an arbitrary family of analytic sets in $O$ may not be analytic. As we will see below, there are strict restrictions for a set to be an analytic set (see Theorem A); for instance, the union of an infinite countable family of circles in $\mathbb{R}^2$ which pairwise meet in the origin cannot be analytic (and such a family of circles can be easily chosen with all the circles being analytic sets). Nevertheless, the following is proved in [84, p. 154]:

**Theorem 1.45.** *If $\mathcal{F}$ is a locally finite family of analytic sets in $O$, then the union of the sets from $\mathcal{F}$ is also an analytic set in $O$.*

Throughout this dissertation we will be frequently interested in extending analytic maps $f : O \to \mathbb{R}$ to the whole surface $S$ keeping at least $C^\omega$ regularity in $O$ and $C^\infty$ regularity in $S \setminus O$. The work is done by Theorem 1.10 in Section 1.2.1. For example, if $S = \mathbb{S}^2$, it is be enough to apply Theorem 1.10 to the map $F : U \to \mathbb{R}$ given by $F(u) = f(u/\|u\|)$ for every $u \in U = \{v \in \mathbb{R}^3 \setminus \{(0,0,0)\} : v/\|v\| \in \mathbb{S}^2\}$. In general, it suffices to recall that any analytic surface $S$ can be analytically embedded into some $\mathbb{R}^m$ and thus its associated tangent vector spaces can be seen as subsets of $\mathbb{R}^m$ (with $m$ depending on the surface) — see [52, Section 6.4] for the details. The combination of this latter fact with Theorem 1.10 implies:

**Theorem 1.46.** *Given any analytic map $F : O \to \mathbb{R}$ defined on a region $O$ of an analytic surface $S$, there exists a $C^\infty$ map $\rho : S \to (0, \infty)$, which is analytic at $O$ and vanishes outside $O$, such that $\rho F$ (after being extended as zero outside $O$) is of class $C^\infty$ on the whole $S$.*

### 1.4.3 Flows associated with vector fields

Many of the more basic and important results in *qualitative theory of planar ordinary differential equations* transfers naturally to surfaces, via Remark 1.36.

Some good references where the qualitative theory of ordinary differential equations is well studied are, among others, [78], [56], [19], [66], [68], [5] or [22]. We devote this section to highlight the translation of some of the main results from this theory to surfaces.

Let $r \geq 1$, $r = \infty$ or $r = \omega$. Local flows are associated, in a natural way, with vector fields (and then to autonomous systems of differential equations via Remark 1.36) defined on $C^r$ surfaces $S$. Namely, if $\Phi : \Lambda \subset \mathbb{R} \times S \to S$ is a local flow (on $S$) which is of class $C^r$ as map, then the map $X : S \to TS$ given by $X(p) = \frac{\partial \Phi}{\partial t}(0, p)$ (the *associated vector field* with $\Phi$) is a $C^{r-1}$ vector field on $S$ and satisfies $\frac{\partial \Phi}{\partial t}(t, p) = X(\Phi(t, p))$ for every $t \in I_p$, that is, $\Phi_p$ is an integral curve of $X$ starting at $p$ (where by $C^{\infty-1}$ and $C^{\omega-1}$ we are denoting, respectively, $C^\infty$ and $C^\omega$). Conversely, if $X : S \to TS$ is a $C^r$ vector field, and $\gamma_p(t)$ denotes the maximal integral curve of $X$ staring at $p$, then $\Phi(t, p) = \gamma_p(t)$ is a local flow on $S$ of class $C^r$.

*Remark* 1.47. When $S = \mathbb{R}^2$ we typically use the identification of $T\mathbb{R}^2$ with $\mathbb{R}^2$. Consequently, by a $C^r$ vector field on $\mathbb{R}^2$ we will simply mean a vector map $f : \mathbb{R}^2 \to \mathbb{R}^2$ of class $C^r$ (sometimes, instead of using the vector notation $f = (f_1, f_2)$, we will write $f = f_1 \partial_x + f_2 \partial_y$ to emphasize the role of vector field played by $f$).

Some specific flows will be mentioned along this dissertation following this identification. Let $f_s$, $f_a$, and $f_r$ be the planar vector fields given by $f_s(x, y) = (1, 0)$, $f_a(x, y) =$

$(-y, x)$ and $f_r(x, y) = (-x, -y)$, and associate to them the corresponding planar flows $\Phi_s, \Phi_a$ and $\Phi_r$. Also, let $f_{v_1}(x, y) = (x^2, 0)$, $f_{v_2}(x, y) = (x, 0)$, $f_{v_3}(x, y) = (-x, 0)$, $f_{h_1}(x, y) = (y^2, 0)$ and $f_{h_2}(x, y) = (y, 0)$, being $\Phi_{v_1}$, $\Phi_{v_2}$, $\Phi_{v_3}$, $\Phi_{h_1}$ and $\Phi_{h_2}$, respectively, their associated planar flows.

Let $(S, \Phi)$ be local flow and $p$ be a singular point of $\Phi$. We say that it is *vertical* (respectively, *horizontal*) if there is a local topological equivalence between $\Phi$ and either $\Phi_{v_1}$, $\Phi_{v_2}$ or $\Phi_{v_3}$ (respectively, $\Phi_{h_1}$ or $\Phi_{h_2}$) at $p$ and $\mathbf{0} = (0, 0)$. A singular point which is neither vertical, nor horizontal, is called *essential*. Among the essential singular points we distinguish the subset of *trivial* ones as those points which admit a neighbourhood of singular points.

*Remark* 1.48. Let $S$ be a $C^r$ surface with $r = \infty$ (respectively $r = \omega$) and $(G, +)$ a countable group and $A : G \times S \to S$ a smooth (respectively analytic) free proper action on $S$ (see Section 1.4.1). Let $X$ be a $C^r$ vector field on $S$, $\Phi$ be its associated local flow and assume that for every $p \in S$ and every $g \in G$ the equality $X(p) = X(A(g, p))$ holds. Given any $p \in S$, denotes by $[p]$ the equivalence class of $p$ (i.e., $[p] = \{A(g, p) : g \in G\}$). It is then easy to prove that the map $\Psi$ defined by $\Psi(t, [p]) := \Phi(t, p)$ is a local flow on $S/G$. For example, let us fix an irrational real number $\alpha \in (0, 1)$ and consider the planar vector field $f_\alpha(x, y) = (1, \alpha)$. After identifying points $(x, y)$ and $(x', y')$ in $\mathbb{R}^2$ when both $x - x'$ and $y - y'$ are integers, the vector fields $f_s$ and $f_\alpha$ induce respectively flows $\Phi_{ss}$ and $\Phi_\alpha$ on the torus $\mathbb{T}^2$. Any flow $(S, \Phi)$ topologically equivalent to $(\mathbb{T}^2, \Phi_{ss})$ (respectively to $(\mathbb{T}^2, \Phi_\alpha)$) is called a *torus rational flow* (respectively a *torus irrational flow*).

Let us fix for the rest of section a $C^r$ surface (for some $r \geq 0$ or $r = \omega$), $S$.

If the local flow associated with a $C^1$ vector field on $S$ is a global flow, then we say that the vector field is *complete*. For example, any $C^1$ vector field on a compact surface is complete [55, Theorem 9.16, p. 216]. On the other hand, if $S$ is a noncompact surface, the flow $\Phi$ associated with a vector field is in general only a local flow. However, one can always work with an well-associated (global) flow in the following terms. Since $S$ is locally compact metric space, we can consider $S_\infty$ the one-point compactification of $S$ which is itself a metric space — see Theorem 1.2. By compactness, all the distances on $S_\infty$ are equivalent; we fix one of those distance and denote it by $d_\infty$. Now, associated with any local flow $(S, \Phi)$, there exists a (global) flow on $S_\infty$, $\Phi_\infty$, having $\infty$ as a singular point, whose restriction $(S, \Phi_\infty)$ is a flow topologically equivalent to $(S, \Phi)$. In this dissertation, the typical case is $S = \mathbb{R}^2$. In this case $S_\infty$ is homeomorphic to $\mathbb{S}^2$ and the flow $(S, \Phi_\infty)$ can be supposed to be of class $C^\infty$. This last facts follows from the combination of Theorem 1.49 below with the fact that the only recurrent orbits of a flow on $\mathbb{S}^2$ are the singular points and the periodic orbits [7, Lemma 2.5, pp. 56].

Let $\Omega \subset S$ be a nonempty compact invariant set of $\Phi$. If $\Omega$ contains no compact nonempty proper subsets invariant for $\Phi$, $\Omega$ is called a *minimal set* (of $\Phi$). A minimal set

$\Omega \subset S$ is said to be *trivial* if it is either a periodic orbit or a singular point or else the whole surface $S$, provided that $(\Omega = S, \Phi)$ is a torus irrational flow. In [38], C. Gutierrez established the following important result:

**Theorem 1.49** (Smoothing Theorem)**.** *Assume that $S$ is a compact $C^\infty$ surface and let $\Phi : \mathbb{R} \times S \to S$ be a flow. Then there exists a $C^1$ flow $\Psi$ on $S$ which is topologically equivalent to $\Phi$. Furthermore, $\Phi$ is topologically equivalent to a $C^\infty$ flow on $S$ if and only if every minimal set of $\Phi$ is trivial.*

We say that an orbit $\varphi(p)$ (seen as an orbit of $(S_\infty, \Phi_\infty)$) is *positively stable* (respectively *negatively stable*) if for any $\epsilon > 0$ there is a number $\delta > 0$ such that $d_\infty(p, q) < \delta$ implies that all points from $\varphi(q, +)$ (respectively $\varphi(-, q)$) stay at a distance less than $\epsilon$ from $\varphi(p, +)$ (respectively $\varphi(-, p)$). An orbit is said to be *stable* if it is both positively and negatively stable, and it is called *unstable* otherwise.

Let $(S, \Phi)$ be a local flow on $S$ and $p \in S$. If there is a local topological equivalence between $\Phi$ and $\Phi_s$ at $p$ and $\mathbf{0}$ (recall definitions in Remark 1.47), with corresponding homeomorphism $h : U_1 \to U_2 = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$, then we say that $U_1$ is a *tubular neighbourhood* (or a *flow box*) of $p$ and call $h^{-1}(\{0\} \times (-\epsilon, \epsilon))$ a *transversal* to $p$ for $\Phi$ (or just a transversal to $\Phi$ —or simply a transversal— when no emphasis on $p$ is required). It is a well-known fact that every regular point $p$ of $\Phi$ admits a tubular neighbourhood (this is the so-called *Flow Box Theorem*, see, e.g., [13, Theorem 2.9, p. 50]). Moreover, the following stronger result also holds for regular points (see, e.g., [13, Theorem 2.11, p. 51]):

**Lemma 1.50** (Long Flow Box Theorem)**.** *Let $p$ be a regular point of $\Phi$ and $\Gamma \subset \varphi(p)$ be a compact semiorbit which is not a circle. Then there exists a tubular neighbourhood of $p$, $U_p$, such that $\Gamma \subset U_1$.*

A set $M \subset S$ is said to be a *lateral tubular region* (or a *semi-flow box*) at $p$ if there is a homeomorphism $h : M \to [-1, 1] \times [0, 1]$ such that $h^{-1}([-1, 1] \times \{s\})$ is a semiorbit of $\Phi$ for every $s \in (0, 1]$ and $p = h^{-1}((0, 0))$. We call the arc $h^{-1}([-1, 1] \times \{0\})$ (respectively the semi-open arc $h^{-1}(\{0\} \times (0, 1]))$ the *border* of the semi-flow box $M$ (respectively *lateral transversal* at $p$). We also say that $h^{-1}((-1, 1) \times [0, 1))$ is a *semi-open flow box*. The continuity of $\Phi$ implies that, although the border of a semi-flow box need not be a semiorbit of $\Phi$, it is the union of some of its semiorbits. For example, if $p$ is a horizontal singular point, there exists a neighbourhood of $p$ which can be decomposed as the union of two semi-flow boxes which meet exactly at their border.

If $\mu : \mathbb{R} \to S$ is a continuous injective map with the property that, for any $s \in \mathbb{R}$, there is $\epsilon_s > 0$ such that $\mu((s - \epsilon_s, s + \epsilon_s))$ is a transversal to $\mu(s)$, then we call $\mu(\mathbb{R})$ a *transversal* to the flow $(S, \Phi)$.

Let $r \geq 1$, $r = \infty$ or $r = \omega$. When we are dealing not only with a (continuous) local flow but with a $C^r$ local flow (for some $r \geq 2$, $r = \infty$ or $r = \omega$), $\Phi$, and its associated

$C^{r-1}$ vector field (where we understand $\infty - 1 = \infty$ and $\omega - 1 = \omega$), $X$, we can also work with a *differentiable* version of the Flow Box Theorem. We first give a restricted notion of transversal in this context.

A $C^r$ embedding $\lambda : I \to S$ of an open interval $I \subset \mathbb{R}$ is called a $C^r$ *transversal section* for $X$ if the vectors $\lambda'(s)$ and $X(\lambda(s))$ are linearly independent for any $s \in I$. For example, if $S = \mathbb{R}^2$, $w \in \mathbb{R}^2$ is a regular point of $X$ and $X(w)$ is linearly independent to $v \in \mathbb{R}^2$, then $\lambda : (-\epsilon, \epsilon) \to \mathbb{R}^2$ defined by $\lambda(s) = w + sv$ is an analytic transversal section provided that $\epsilon > 0$ is small enough.

Let $X_1 : S_1 \to TS_1, X_2 : S_2 \to TS_2$ be two $C^r$ vector fields and let $\Phi_1 : \Delta_1 \to S_1$ and $\Phi_2 : \Delta_2 \to S_2$ be their associated flows. We say that $X_1$ is $C^r$ *conjugate* to $X_2$ if there is a $C^r$ diffeomorphism $h : S_1 \to S_2$ such that the domains $I_{1,p}$ for $\Phi_{1,p}$ and $I_{2,h(p)}$ for $\Phi_{2,h(z)}$ are equal and $h(\Phi_1(t,p)) = \Phi_2(t, h(p))$ for every $(t,p) \in \Delta_1$.

**Theorem 1.51** ($C^r$ Flow Box Theorem)**.** *Let $X$ be a $C^r$ vector field on a $C^r$ surface $S$ (for some $1 \leq r \leq \infty$ or $r = \omega$). Let $\lambda : I \to S$ be a $C^r$ transversal section for $X$, assume that $[c,d] \subset I$ for some $c < 0 < d$, and write $\lambda(0) = w$. Then there exist $\epsilon > 0$, an open neighbourhood $W$ of $w$ in $U$ and a $C^r$ diffeomorphism $h : W \to (-\epsilon, \epsilon) \times (c,d)$ such that:*

*(i) $\lambda(I) \cap W = \lambda((c,d))$ and $h(\lambda(s)) = (0,s)$ for any $s \in (c,d)$;*

*(ii) $h$ is a $C^r$ conjugacy between $X|_W$ and the constant vector field $Y : (-\epsilon, \epsilon) \times (c,d) \to \mathbb{R}^2$ given by $Y(x,y) = (1,0)$.*

*Proof.* See for example [22, Theorem 1.12]. □

*Remark* 1.52. ([7, Theorem 1.1, p. 45]). In the same setting as in the previous result, let $\Phi$ be the local flow associated with $X$. Let $z \in S$ be a regular point whose orbit is not periodic, let $a < b$ be two points in $I_z$ and consider the semiorbit $\Phi_z([a,b])$. Then, using compactness and the previous theorem, it is possible to find a neighbourhood $W$ of $\Phi_z([a,b])$ in $S$ and a small $\epsilon > 0$ such that $X$ is $C^r$ conjugate in $W$ to the constant vector field $Y : (a - \epsilon, b + \epsilon) \times (-1,1) \to \mathbb{R}^2$ given by $Y(x,y) = (1,0)$.

### Some results for the special case of the plane and the sphere

One of the landmarks of bidimensional qualitative theory of differential equations is the famous Poincaré-Bendixson Theorem. This theorem deals with the asymptotic behaviour of the orbits of a flow, that is, with the study of the $\alpha$-limit and $\omega$-limit sets of a flow. In general, $\omega$-limit sets (respectively $\alpha$-limit sets) are closed subsets of $S$ invariant for $\Phi$. If the $\omega$-limit set of $p$ is nonempty, then $b_p = \infty$; moreover, when $S$ is compact, $\omega_\Phi(p)$ is also compact and connected. The analogous statements also hold for $\alpha$-limit sets [66, Chapter 1].

A deeper result for the case of local flows on open subsets of the plane is given by the classical Poincaré-Bendixson Theorem. Let us fix for the rest of the section a region $U \subset \mathbb{R}^2$, a vector field on $U$ of class $C^r$ with $1 \leq r \leq \infty$ or $r = \omega$ and its associated local flow $\Phi$.

**Theorem 1.53** (Poincaré-Bendixson Theorem)**.** *Let $p \in U$ and suppose that the semiorbit $\varphi(p, +)$ (respectively $\varphi(-, p)$) is contained in a compact subset of $U$. Then either the set $\omega_\Phi(p)$ (respectively $\alpha_\Phi(p)$) contains some singular point or it is a periodic orbit.*

*Proof.* See, for example, [22, Theorem 1.25, p. 24] or [5, Theorem 13, p. 92]. □

The following classical corollary of the Poincaré-Bendixson Theorem is of special interest:

**Theorem 1.54.** *If $\Gamma$ is a periodic orbit of $\Phi$ enclosing (as a circle on $\mathbb{R}^2$) a simply connected region $B$ which is totally contained in $U$, then $B$ contains a singular point of $\Phi$.*

*Proof.* See [22, Theorem 1.31] or [5, Theorem 16, p. 97]. □

In standard proofs of the Poincaré-Bendixson Theorem the result below is stated as a preliminary lemma. Its proof can be found for example in [22, Lemma 1.29] or [5, Theorem 11, p. 90].

**Theorem 1.55.** *Let $p \in U$ and assume that $\varphi(p, +)$ (respectively, $\varphi(-, p)$) is contained in a compact subset of $U$. If the $\omega$-limit or the $\alpha$-limit set of an orbit $\Gamma \subset \omega_\Phi(p)$ (respectively, $\Gamma \subset \alpha_\Phi(p)$) contains some regular point, then $\Gamma$ is periodic and $\omega_\Phi(p) = \Gamma$ (respectively $\alpha_\Phi(p) = \Gamma$).*

We finish this section with an elementary and technical lemma.

**Lemma 1.56.** *Let $k : U \to \mathbb{R}$ be a $C^r$ function and consider the vector field $Y = kX$. Let $p$ be a regular point of $Y$ and $\Psi_p : J_p \to U$ be the maximal integral curve of $Y$ starting at $p$. Then there exist an open interval $0 \in L \subset \mathbb{R}$ and a $C^r$ function $\tau : L \to \mathbb{R}$ such that $\tau(0) = 0$ and $\Phi_p(t) = \Psi_p(\tau(t))$ for all $t \in L$.*

*Proof.* If $p$ is a regular point for $Y$, it cannot be a zero of $k$. Therefore, there is an open interval $0 \in J \subset I_p$ where the function $F = 1/(k \circ \Psi_p)$ is well-defined and of class $C^r$.

Let us consider now the maximal solution of Cauchy's problem

$$\begin{cases} \dot{\tau} = F(\tau) \\ \tau(0) = 0. \end{cases}$$

This maximal solution is a function $\tau$ defined in an open interval $L$ containing 0 and whose evaluations belongs to $J$. It is easy to check that the composition $\Psi_p \circ \tau$ is an integral curve of $X$ starting at $p$. Hence we have $L \subset I_p$ and $\Phi_p(t) = \Psi_p(\tau(t))$ for all $t \in L$. $\square$

### Bendixson Compactification

For the sake of simplicity, we present an adaptation of [72, Section 1.1.3.2] using real analysis notation.

The one-point compactification of the euclidean plane $\mathbb{R}^2_\infty = \mathbb{R}^2 \cup \{\infty\}$ can be seen as a analytic compact surface. Indeed, it is enough to consider the two local charts $(\mathbb{R}^2, z)$ and $(\mathbb{R}^2_\infty \setminus \{0\}, Z)$ where $z : \mathbb{R}^2 \to \mathbb{R}^2$ is the map given by the formula $z(x,y) = (r(x,y), s(x,y)) = (x,y)$ for every $(x,y) \in \mathbb{R}^2$ and $Z : \mathbb{R}^2_\infty \setminus \{0\} \to \mathbb{R}^2$ is given by $Z(x,y) = (u(x,y), v(x,y)) = (x/(x^2+y^2), -y/(x^2+y^2))$ if $(x,y) \neq \infty$ and $Z(\infty) = (u(\infty), v(\infty)) = 0$. The equations for the changes of coordinates $z \circ Z^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ and $Z \circ z^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ are given by the analytic formulas $r = u/(u^2+v^2)$ and $s = -v/(u^2+v^2)$ and $u = r/(r^2+s^2)$ and $v = -s/(r^2+s^2)$ respectively; this justifies that $\{(\mathbb{R}^2, z), (\mathbb{R}^2_\infty \setminus \{0\}, Z)\}$ is an analytic atlas for $\mathbb{R}^2_\infty$. We denote by $\phi : \mathbb{R}^2_\infty \to \mathbb{R}^2_\infty$ the homeomorphism associated with this transition map (where $\phi(0) = \infty$ and $\phi(\infty) = 0$); we will call $\phi$ the *transition homeomorphism associated with the Bendixson compactification.*

The spaces $\mathbb{R}^2_\infty$ and $\mathbb{S}^2$ are not only homeomorphic but also analytically diffeomorphic: as a explicit analytic diffeomorphism we may consider the map $\psi : \mathbb{S}^2 \to \mathbb{R}^2_\infty$ given by the formulas $\psi(0,0,1) = \infty$ and $\psi(x,y,z) = (x/(1-z), y/(1-z))$ if $(x,y,z) \neq (0,0,1)$. The standard euclidean distance on $\mathbb{R}^3$, $d_2$, induces a distance on $\mathbb{R}^2_\infty$ (compatible with its topology as one-point compactification): the map given by $d_\infty(a,b) = d_2(\psi^{-1}(a), \psi^{-1}(b))$ for every $a, b \in \mathbb{R}^2_\infty$ is such a distance.

Let $P$ and $Q$ be real polynomials in two variables and consider the algebraic planar vector field given by $X = P\partial_x + Q\partial_y$. If $d = \max\{\deg(P), \deg(Q)\}$, we may consider a vector field in the amplified euclidean plane $\mathbb{R}^2_\infty$, $\hat{X}$, given by the formulas $\hat{X}(r,s) = \frac{1}{1+(r^2+s^2)^d}\left(P(r,s)\partial_r + Q(r,s)\partial_s\right)$ if $(r,s) \in \mathbb{R}^2_\infty \setminus \{\infty\}$ and $\hat{X}(\infty) = 0$. It is direct to show that $\hat{X}$ is well-defined and analytic in the whole $\mathbb{R}^2_\infty$.

### Finite sectorial decomposition property

Let $\Phi$ be a global flow on $\mathbb{R}^2$ having the origin $\mathbf{0} = (0,0)$ as an isolated singular point, that is, such that there exists an open neighbourhood of $\mathbf{0}$, $U$, where the only singular point of $\Phi$ is $\mathbf{0}$.

A *characteristic orbit* $\varphi(p)$ at $\mathbf{0}$ is a regular orbit tending to $\mathbf{0}$ in positive time (respectively in negative time) with a well-defined slope, that is, $\lim_{t\to\infty} \Phi_p(t) = \mathbf{0}$ and the

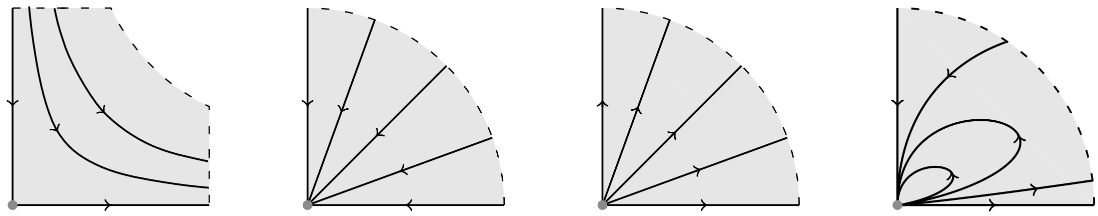

Figure 1.1: From left to right: a hyperbolic, an attracting, a repelling and an elliptic sector.

limit $\lim_{t\to\infty}(\Phi_p(t)-p)/\|\Phi_p(t)-p\|$ exists (respectively $\lim_{t\to-\infty}\Phi_p(t)=\mathbf{0}$ and the limit $\lim_{t\to-\infty}(\Phi_p(t)-p)/\|\Phi_p(t)-p\|$ exists).

Let $f_i:\mathbb{R}^2\to\mathbb{R}^2$, $1\le i\le 4$, be the vector fields $f_1(x,y)=(x,-y)$, $f_2(x,y)=(-x,-y)$, $f_3(x,y)=(x,y)$, $f_4(x,y)=(x^2-2xy,xy-y^2)$ respectively. Also, let

$$A_1=\{(x,y)\in\mathbb{R}^2:0\le x,y<1,xy<1/2\},$$

$$A_2=A_3=A_4=\{(x,y)\in\mathbb{R}^2:0\le x,y<1,x^2+y^2<1\}.$$

We remark that although the sets $A_i$ are not open, $f_i$ still induces a local flow $\Phi_i$ on $A_i$, $1\le i\le 4$. See Figure 1.1. Assume now that $B$ is a set containing $\mathbf{0}$ and $\Phi$ induces a local flow on $B$ which is topologically equivalent to $\Phi_i$. Then we say that $B$ is a *hyperbolic*, *attracting*, *repelling* or *elliptic sector* of $\Phi$ (at $\mathbf{0}$) when, respectively, $i=1,2,3,4$ (in the cases $i=2$ and $i=3$ we also say that $B$ is a *parabolic sector*). The flow $\Phi$ is said to have the *finite sectorial decomposition property* (at $\mathbf{0}$) if $\mathbf{0}$ has a bounded neighbourhood $V$ such that there are no periodic orbits on $V$ and there are finitely many characteristic orbits at $\mathbf{0}$, $c_0,c_1,\dots,c_{n-1}$, which meet $\operatorname{Bd}V$ transversely at points $p_0,p_1,\dots,p_{n-1}$ (in the sense that each $p_i$ has a neighbourhood in $\operatorname{Bd}V$ which is transversal for $\Phi$) decomposing $V$ in the union of $n$ hyperbolic, attracting, repelling and elliptic sectors.

*Remark* 1.57. The typical case for this to happen is that $\Phi$ is associated with an analytic vector field analytic with $\mathbf{0}$ as isolated singularity. It then can be proved that either $\mathbf{0}$ is a focus (that is, a global attractor or repeller such that none of the orbits in any neighbourhood of $\mathbf{0}$ are characteristic), a center or $\Phi$ has the finite sectorial decomposition property at $\mathbf{0}$ — see for instance [22, Chapter 3]. The proof of this fact in [22] (or in [42]) uses highly nontrivial and sophisticated desingularization methods. Fortunately, in most of the cases where we are interested in using this property, we will have the extra hypothesis of absence of periodic orbits in a small neighbourhood of $\mathbf{0}$. In that context, a elementary alternative proof is given in [56, Chapter X] as a combination of the Star Structure Theorem (Theorem A) with some standard Poincaré-Bendixson techniques.

# Chapter 2

# The Star Structure Theorem

The local structure of the set of zeros of analytic functions on the plane will play a central role during the dissertation. Before presenting this structure, we introduce the auxiliary notion of star.

Given any positive integer $n \in \mathbb{N}$, we say that a topological space is an *$n$-star* if it is homeomorphic to $S_n = \{z \in \mathbb{C} : z^n \in [0,1]\}$. If $Z$ is an $n$-star and $h : S_n \to X$ is a homeomorphism, then the image of the origin (respectively the image of any of the $n$-roots of unity) under $h$ is called a *center of the star* (respectively an *endpoint of the star*) while the components of $Z \setminus \{h(0)\}$ are called the *branches of the star*. Note that the center, the endpoints and the branches of a star are uniquely defined except in the cases $n = 1, 2$, when $Z$ is just an arc and the centers are its endpoints (for $n = 1$) or its interior points (for $n = 2$). We will also adopt the convention of calling any singleton a 0-star (the point being its center). When $Y$ is a topological space and $a$ is a point in $Y$ which possesses a neighbourhood $Z \subset Y$ being an *$n$-star* with $a$ as center, we will say that $a$ is a *star point in $Y$ (of order $n$)*; note that the order of a star point is unambiguously defined. If all the points in $Y$ are star points, $Y$ will be called a *generalized graph*.

The local structure of planar analytic functions is stated in the following result. The contents of the theorem are classical and well-known, see, e.g., [43, Theorem 4.3], except maybe for the "parity" statement, which is due to Sullivan [79]. Alternatively, in [25] we presented a "dynamically based" proof. The proof we expose in Subsection 2.1 below is the latter one.

**Theorem A** (Star Structure Theorem). *Let $U \subset \mathbb{R}^2$ be a region and $f : U \to \mathbb{R}$ be an analytic function. Let $C = \{z \in U : f(z) = 0\}$ be the set of zeros of $f$. Then either $C = U$*

*or given any non-isolated point of $C$, $z \in C$, there exists a neighbourhood $V$ of $z$ and an $n \in \mathbb{N}$ such that $V \cap C$ is a $2n$-star with center $z$. Moreover, in the latter case, after removing from the star its center and endpoints, the resultant open arcs admit analytic parametrizations.*

*Remark* 2.1. In fact, as shown implicitly in [43], a stronger result also holds (which cannot be derived from our proof). With the notation of the theorem above, and in the case when $V \cap C$ is a $2n$-star with center $z$, the arc $L$ given by the union of any branch with $\{z\}$ can be also parametrized by an analytic map $\varphi : [0,1] \to L$ (where by $\varphi$ being analytic in the whole closed interval we mean that it can be analytically extended to a larger open interval containing $[0,1]$).

## 2.1 Proof of Theorem A

Let $U \subset \mathbb{R}^2$ be open and connected, $f : U \to \mathbb{R}$ be an analytic function and $C = \{z \in U : f(z) = 0\}$.

If $C$ has nonempty interior (as a subset of $U$), then $f$ is identically zero (recall Proposition 1.7) and there is nothing to prove. In what follows we assume that $\mathrm{Int}(C) = \emptyset$.

Let $z_0 \in U$ be a non-isolated point of $C$. We will build an analytic vector field $X$ on a specific open neighbourhood $W$ of $z_0$ such that either all its points are regular points for $X$ (and then we will see that, in a neighbourhood of $z_0$, $C$ reduces to an arc with $z_0$ in its interior), or $z_0$ is the only singular point of $X$ in $W$ and there is a compact neighbourhood of $z_0$, $V \subset W$, which can be written as a finite union of evenly many sectors, with the additional property that $C \cap V$ is the union of $\{z_0\}$ and the orbits separating the adjoining sectors.

To define this vector field we proceed in two steps. Firstly, we consider the planar vector field $Y$ given by $Y(z) = (-\frac{\partial f}{\partial y}(z), \frac{\partial f}{\partial x}(z))$. Secondly, by virtue of Theorem 1.9, there exist a neighbourhood of $z_0$, $W \subset U$, and analytic functions $k, X_1, X_2 : W \to \mathbb{R}$ such that if we call $X = (X_1, X_2)$, then $Y = kX$ and $X$ has no zeros in $W \setminus \{z_0\}$. This $X$ is the vector field we are looking for. We will denote by $\Phi$ the local flow associated with $X$.

Notice that, replacing $f$ by $f^2$ if needed, there is no loss of generality in assuming that $\frac{\partial f}{\partial x}(z) = \frac{\partial f}{\partial y}(z) = 0$ for all $z \in C$. Moreover, $f$ is a *first integral* for $X$, that is, if $z : J \subset \mathbb{R} \to W$ is an integral curve of $X$, then $f \circ z$ is constant. Indeed, if $t \in J$ is such that $z(t)$ is a singular point of $X$, then both partial derivatives of first order of $f$ vanish at $z(t)$; otherwise we apply Lemma 1.56 to guarantee that $\dot{z}(t)$ and $Y(z(t))$ are proportional vectors. Therefore, we get in any case that

$$\frac{d(f \circ z)}{dt}(t) = \left\langle \left( \frac{\partial f}{\partial x}(z(t)), \frac{\partial f}{\partial y}(z(t)) \right), \dot{z}(t) \right\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of $\mathbb{R}^2$.

Obviously there are two options for $z_0$: either it is an isolated singular point of $X$ or it is a regular one. We will distinguish these two cases in the following reasoning.

We first handle the case when all points of $W$ are regular for $X$. According to the $C^\omega$ Flow Box Theorem (Theorem 1.51), $W$ can be chosen in such a way that all the orbits of $X$ accumulate at the boundary of $W$ and intersect (the image of) an analytic transversal section $\lambda : J \to W$ for $X$ (with $0 \in J$ and $\lambda(0) = z_0$) at exactly one point. Let $s \in J$ and consider the maximal integral curve of $X$ starting at $z_s = \lambda(s)$, $\Phi_{z_s}(t)$. The composition $f \circ \Phi_{z_s}$ is constant; when $s = 0$ that constant is necessarily zero. Taking into account Proposition 1.8 and the fact that $C$ has empty interior, we get that $f \circ \Phi_{z_s}$ do not vanish if $s \in J \setminus \{0\}$ is close enough to 0. Thus, choosing if necessary a smaller neighbourhood $W$ of $z_0$, we get that the zeros of $f$ in $W$ are exactly the points of the orbit $\varphi_\Phi(z_0)$.

Now we consider the case when $z_0$ is an isolated singularity of $X$ (in fact, because of the way we have defined $W$, it is the only singularity of $X$). Since $f$ is a first integral, if the $\alpha$-limit set or the $\omega$-limit set of an integral curve $z(t)$ of $X$ contains $z_0$, then the function $f \circ z$ must be identically zero. As a consequence, $\{z_0\}$ cannot be, at the same time, the $\alpha$-limit and the $\omega$-limit set of any integral curve $z(t)$ of $X$, that is, the system $X$ admits no homoclinic orbits (except $\{z_0\}$ itself). Indeed, if we suppose the contrary, then the image of $z(t)$ (together with $z_0$) defines a circle. Therefore, the Poincaré-Bendixson theorem and Theorem 1.54 imply that all orbits in the region enclosed by this circle are homoclinic as well and $C$ has nonempty interior, a contradiction.

We claim that $X$ admits no sequences of periodic orbits $(J_n)_{n\in\mathbb{N}}$ satisfying $J_n \subset B(z_0, 1/n) \cap W$ for all $n$. We argue to a contradiction by assuming that such a sequence does exist. Recall that any periodic orbit in $W$ encloses $z_0$ (by Theorem 1.54), so given any two of them one encloses the other; in particular we can assume that $J_n$ encloses $J_{n+1}$ for every $n$. Besides, since $z_0$ is not an isolated zero of $f$, given an arbitrary $n$ one finds a $z \neq z_0$ in the region enclosed by $J_n$ such that $f(z) = 0$. Say that $z$ belongs to the annulus between consecutive curves $J_m$ and $J_{m+1}$, $m \geq n$. By the Poincaré-Bendixson Theorem (Theorem 1.53), two possibilities arise for the orbit of $z$: either it is a periodic orbit consisting of zeros of $f$ or it spirals towards two periodic orbits (both consisting of zeros of $f$) included in the fixed annulus. Therefore, one can also consider a new sequence of periodic orbits $(J'_n)_{n\in\mathbb{N}}$ such that each $J'_n$ consists of zeros of $f$ and verifies that $J_n$ encloses $J'_n$ and $J'_n$ encloses $J'_{n+1}$. Consequently, the analytic function $\tau \mapsto f(z_0 + \tau(1,0))$ vanishes at a sequence of points $(\tau_n)_n$ converging to 0 so, by Proposition 1.8, it vanishes in a full open interval containing 0, say $(-\delta, \delta)$. Now realize that any orbit of any point near enough to $z_0$ must either be periodic or spiral around $z_0$, hence it must intersect the segment $\{z_0 + \tau(1,0) : \tau \in (-\delta, \delta)\}$. We conclude that $f$ vanishes in a neighbourhood of $z_0$, contradicting that $C$ has empty interior.

As a consequence of the above claim, there is a small neighbourhood $W'$ of $z_0$ such that the vector field $X$, when restricted to $W'$, has no periodic orbits. (Note that this does not exclude the possibility that the initial system has a sequence of periodic orbits accumulating at $z_0$; however, since their diameters cannot be too small, they become non-periodic after being "cut off" by $W'$.) To ease the notation we still call the new neighbourhood and the corresponding vector field $W$ and $X$, respectively.

Next, let us consider a small enough $r > 0$ such that $V = \mathrm{Cl}(B(z_0, r)) \subset W$. In the absence of homoclinic and periodic orbits, taking into account that $z_0$ is the only singular point of $X$, and using the Poincaré-Bendixson Theorem and Theorem 1.55, the semiorbits of the system in $V$ can be easily described. Namely, if $z \in \mathrm{Int}(V) \setminus \{z_0\}$ and $\varphi(z)$ is the orbit of $X$ through $z$, then the connected component of $\mathrm{Cl}(V) \cap \varphi(z)$ containing $z$ is either an arc with an endpoint in $\partial V$ or (together with $z_0$) an arc whose endpoints are $z_0$ and a point of $\partial V$. More precisely, let $z \in \mathrm{Int}(V) \setminus \{z_0\}$, let $\Phi_z : (a_z, b_z) \to W$ be the maximal integral curve of $X$ with $\Phi_z(0) = z$, and write $\Phi_z((a_z, b_z)) = \varphi(z)$. Then the component of $\varphi(z) \cap V$ containing $z$ is either $\Phi_z([a'_z, b'_z])$, $\Phi_z((a_z, b'_z])$ (with $a_z = -\infty$), or $\Phi_z([a'_z, b_z))$ (with $b_z = \infty$) for some $a_z < a'_z < 0 < b'_z < b_z$; points $\Phi_z(a'_z), \Phi_z(b'_z)$ belong to $\partial V$; and we have $\lim_{t\to-\infty} \Phi_z(t) = z_0$ and $\lim_{t\to\infty} \Phi_z(t) = z_0$, respectively, in the last two cases.

Among the semiorbits of $X$ in $V$, those having $\{z_0\}$ as their $\alpha$-limit or $\omega$-limit sets consist of zeros of $f$. We claim that there are only finitely many of them. If the opposite is true, then any circle centered in $z_0$ with radius $s$ less than $r$ would contain infinitely many zeros of $f$. Applying Proposition 1.8 to the analytic function $\tau \mapsto f(z_0 + (s\cos\tau, s\sin\tau))$, we get that $f$ vanishes in the whole circle and, since $s$ is arbitrary, in the whole $V$. A similar argument allows us to assume (using if necessary a smaller $r$) that the zeros of $f$ contained in $V$ (apart from $z_0$) are exactly those in the semiorbits having $z_0$ as a limit point. Since $z_0$ is not an isolated point of $C$, the family of these special semiorbits cannot be empty: we denote them by $\Gamma_1, \Gamma_2, \ldots \Gamma_m$ (for some $m \geq 1$) and assume that they are counterclockwise ordered. It only rests to show that $m$ is even.

Let $z_i \in \Gamma_i \cap \partial V$, $i = 1, 2 \ldots, m$. Taking advantage of analyticity once more, there is no loss of generality in assuming that $\partial V$ is locally transversal for $X$ at these points and $\Gamma_i \setminus \{z_i\} \subset \mathrm{Int}(V)$ for any $i$. We call $\Gamma_i$ *outward* or *inward* according to, respectively, $\lim_{t\to-\infty} \Phi_{z_i}(t) = z_0$ or $\lim_{t\to\infty} \Phi_{z_i}(t) = z_0$. We prove that $m$ is even by showing that the semiorbits $\Gamma_i$ are consecutively outward and inward.

Assume, for instance, that $\Gamma_i$ is inward. Let $A$ be the counterclockwise arc in $\partial V$ with endpoints $z_i$ and $z_{i+1}$ (here, we identify $m+1$ and 1; if $m = 1$, then $A = \partial V$). Let $(p_n)_n$ be a sequence of points in $A$ monotonically converging to $z_i$. Since $\partial V$ is locally transversal for $X$ at $z_i$, we can assume that there are semiorbits $\Upsilon_n$ entering $V$ at $p_n$ and escaping from $V$ at corresponding points $q_n \in A$. Observe that the points $q_n$ are reversely ordered as those in the sequence $(p_n)_n$, hence they converge to a point $q \in A$. Let $t_n$ be the escaping time of $\Upsilon_n$, that is $\Phi_{p_n}(t_n) = q_n$. Since $\Phi_{z_i}(t)$ is well defined (and inside

$V$) for any $t \geq 0$, the regularity of the flow at $z_i$ implies $t_n \to \infty$ as $n \to \infty$. Finally, the regularity of the flow at $q$ guarantees that $\Phi_q(t)$ is well defined, and inside $V$, for any $t \leq 0$, that is, $q = z_{i+1}$ and $\Gamma_{i+1}$ is outward (see Remark 1.52). This finishes the proof.

*Remark* 2.2. Note that the two last paragraphs of the above argument can be disposed of: since $W$ contains neither homoclinic nor periodic orbits, and only finitely many heteroclinic orbits, $z_0$ must admit a neighbourhood consisting of a finite number $n$ of hyperbolic sectors. The point $z_0$ is, alternatively, the $\alpha$-limit set and the $\omega$-limit set of the orbits limiting these sectors, hence $n$ is even (instead, we can use the Poincaré index formula [22, Proposition 6.32] (alternatively see Example 1.42) to deduce that the topological index of $z_0$ is the integer $1 - n/2$, so $n$ is even).

Yet, as indicated at Section 1.4.3, the only elementary proof of the finite sectorial decomposition property we are aware of is based on the Star Structure Theorem. Thus, in order to avoid a circular argument, we are bound to (implicitly) use desingularization and, in a sense, the simple profile of our proof is lost.

# Chapter 3

# On the Markus-Neumann Theorem

The Markus-Neumann Theorem is an often cited result dealing with the topological classification of surface flows. Google Scholar provides 147 explicit references to [59], and 91 to [63], and has been used without explicit mention (mainly in the planar case), as in [70, p. 5], a large number of times. A first planar version was proved by L. Markus [59], under the additional restriction of nonexistence of so-called “limit separatrices”. In [63], D. A. Neumann disposed of this condition and extended the result to arbitrary surfaces. Roughly speaking, the theorem states that two surface flows are equivalent if there is a surface homeomorphism preserving a number of distinguished orbits from both flows. However, Markus missed an important point which, apparently, also passed unnoticed to Neumann and the subsequent readers (see for instance [22, pp. 33–34], [68, p. 294], [65, pp. 245–246] or [64, pp. 225–226]). As a consequence the theorem, as stated in [59] and [63], is wrong. In fact, as we will show in the next section, counterexamples can be found in far from pathological settings, even for polynomial plane flows. The good news is that, after appropriately amending the Markus-Neumann notion of separatrix, the theorem works (and can be slightly improved).

We should stress that, when using the theorem in the polynomial scenery, researchers typically employ an alternative, easier to handle with, notion of separatrix (see Remark 3.5). Fortunately enough, it turns out to be equivalent to our amended definition (but not to that of Markus-Neumann’s). Therefore, all such papers remain correct without further changes.

## 3.1 The Markus-Neumann Theorem

In what follows we list a number of notions that will be needed to state the Markus-Neumann theorem (Theorem 3.2 below).

Let $\Phi : \mathbb{R} \times S \to S$ be a flow on a surface $S$. Note that $S$ is not assumed to be either compact nor orientable. Associated with the flow $(S, \Phi)$, fix also the extended flow $(S_\infty, \Phi_\infty)$ (see Section 1.4.3).

Let $\Omega$ be an invariant region for $(S, \Phi)$. Following [63] (and recalling the notation introduced in Remark 1.47), we say that $\Omega$ is is *parallel* when the restriction $(\Omega, \Phi)$ is topologically equivalent to either $(\mathbb{R}^2, \Phi_s)$, $(\mathbb{R}^2 \setminus \{\mathbf{0}\}, \Phi_a)$, $(\mathbb{R}^2 \setminus \{\mathbf{0}\}, \Phi_r)$ or $(\mathbb{T}^2, \Phi_{ss})$, and use the terms *strip*, *annular*, *radial* and *toral*, respectively, to distinguish cases. Note that in the toral case $\Omega = S$ is indeed a torus and all orbits are periodic.

If $\Omega$ is parallel, and $T \subset \Omega$ is a transversal open arc, then we say that it is a *complete* transversal to $\Omega$ provided that one the following conditions hold:

- $\Omega$ is either a strip or an annular region, and $T$ intersects each orbit from $\Omega$ at exactly one point. Observe that, in the strip case, $T$ decomposes $\Omega$ into two regions, $\Omega_T^-$ and $\Omega_T^+$, corresponding to the backward and forward direction of the flow.
- $\Omega$ is a radial region and, if $p \in T$, then each of the two transversals into which $p$ decomposes $T$ intersects any orbit from $\Omega$ infinitely many times.

Also, we say that a transversal $T \subset \Omega$ is *semi-complete* when either it is complete, or it is one of the two transversals into which some point decomposes a complete transversal.

Let $(S, \Phi)$ be a flow and let $p \in S$. Recall that given any orbit $\varphi(p)$ of $\Phi$, we write $\alpha'(p) = \alpha(p) \smallsetminus \varphi(p)$ and $\omega'(p) = \omega(p) \smallsetminus \varphi(p)$. In [63] or [59] separatrices are defined as follows:

**Definition 3.1.** We say that an orbit $\varphi(p)$ of $(S, \Phi)$ is *ordinary* if it is neighboured by a parallel region $\Omega$ such that:

(i) $\alpha'(q) = \alpha'(p)$ and $\omega'(q) = \omega'(p)$ for any $q \in \Omega$;

(ii) $\operatorname{Bd}\Omega$ is the union of $\alpha'(p)$, $\omega'(p)$ and exactly two orbits $\varphi(a)$ and $\varphi(b)$ with $\alpha'(a) = \alpha'(b) = \alpha'(p)$ and $\omega'(a) = \omega'(b) = \omega'(p)$.

If an orbit is not ordinary, then it is called a *separatrix*.

Observe that no conditions are imposed on $\alpha'(p)$ and $\omega'(p)$: one or both may be empty (which is to say the infinite point $\infty$ when passing to $\Phi_\infty$), and they may have, or not, empty intersection. On the other hand, $\varphi(a)$ and $\varphi(b)$ must be distinct and disjoint from $\alpha'(p)$ and $\omega'(p)$.

Let $(S, \Phi)$ be a flow and call $\mathcal{S}$ the union set of all its separatrices. Note that the set $\mathcal{S}$ is closed. The components of $S \setminus \mathcal{S}$ are called the *canonical regions* of $(S, \Phi)$. By a *separatrix configuration* for $(S, \Phi)$, $\mathcal{S}^+$, we mean the union of $\mathcal{S}$ together with a representative orbit from each canonical region.

Let $\Phi_1$ and $\Phi_2$ be two flows defined on the same surface $S$ and let $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$ be, respectively, their separatrix configurations. We say that $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$ are *equivalent* if there is a homeomorphism of $S$ onto $S$ which carries orbits of $\mathcal{S}_1^+$ onto orbits of $\mathcal{S}_2^+$ preserving time directions.

We are ready to state the announced result:

**Theorem 3.2** (Markus-Neumann Theorem [59, 63])**.** *Let $S$ be a surface and suppose that $\Phi_1$ and $\Phi_2$ are flows on $S$ whose sets of singular points are discrete. Then $\Phi_1$ and $\Phi_2$ are topologically equivalent if and only if they have equivalent separatrix configurations.*

## 3.2 Counterexamples to the Markus-Neumann Theorem

As it turns out, Theorem 3.2 (as presently formulated) is wrong, the problem being that the previous definition of separatrix is too restrictive. A planar counterexample is shown by Figure 3.1. Both flows share the orbits $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_5$ and the singular point $p$, and the separatrices are just, in both cases, $p$, $\Gamma_1$ and $\Gamma_2$. For instance, to show that $\Gamma_3$ is ordinary for the right-hand flow $\Phi_2$, take an orbit $\Gamma$ enclosed by $\Gamma_2$ but not by $\Gamma_3$, and use the strip $\Omega$ consisting of all orbits enclosed by $\Gamma_2$ but not by $\Gamma$. Now the boundary of $\Omega$ consist, as required by Definition 3.1, of the orbits $\Gamma_2$ and $\Gamma$, and the singular point $p$, which is both the $\alpha$-limit and the $\omega$-limit set of all orbits in $\Omega$ and also of $\Gamma_2$ and $\Gamma$. (Here, as in the examples below, there is no need to distinguish between $\alpha(q)$ and $\alpha'(q)$ nor between $\omega(q)$ and $\omega'(q)$, because the only recurrent orbits are the singular points). Likewise, $\Gamma_4$ is ordinary for the left-hand flow $\Phi_1$ (use the strip $\Omega$ consisting of all orbits enclosed by $\Gamma_2$ but not by $\Gamma_3$).

Now, since the separatrix configurations $\mathcal{S}_1^+ = \mathcal{S}_2^+ = \{p\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5$ are the same, the flows $\Phi_1$ and $\Phi_2$ should be, according to Theorem 3.2, topologically equivalent. Clearly, they are not: since $\Gamma_2$ is, in both cases, the maximal homoclinic orbit, the topological equivalence should carry it onto itself. However, there are two unstable orbits ($\Gamma_3$ and $\Gamma_4$) inner to $\Gamma_2$ for $\Phi_1$, but just one ($\Gamma_3$) for $\Phi_2$.

*Remark* 3.3. As shown in Chapter 4 (see also [74]) flows $\Phi_1$ and $\Phi_2$ can in fact be realized by polynomial planar vector fields with an only singular point (or, via the Bendixson compactification — Section 1.4.3—, by analytic sphere flows with just two singular points).

An even cleaner (torus) counterexample is exhibited by Figure 3.2. Here, the left-hand flow $\Phi_3$ and the right hand flow $\Phi_4$ share the orbits $\Gamma_1$ and $\Gamma_2$ and the singular point $p$, and

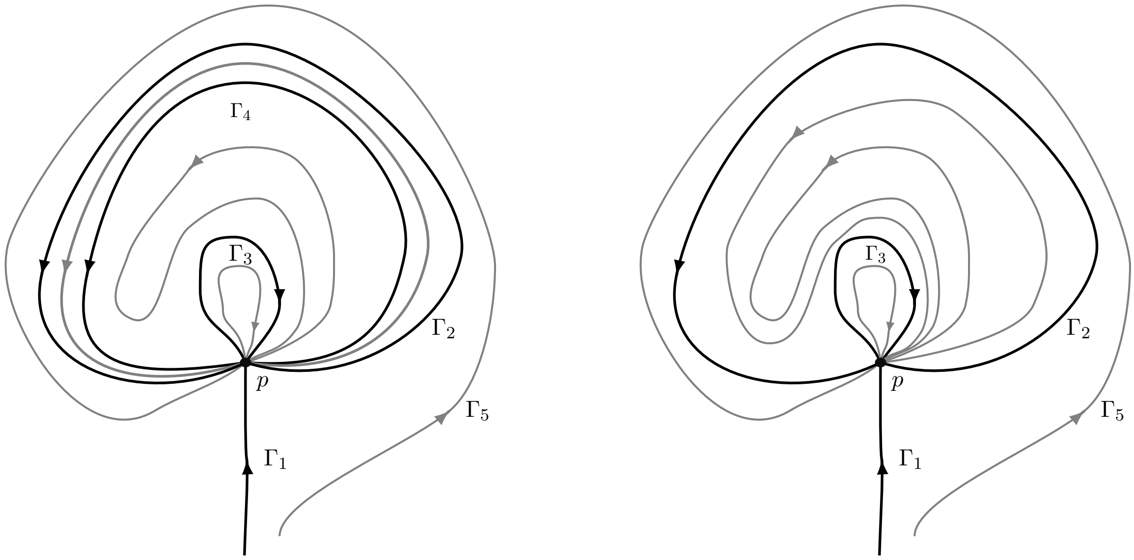


Figure 3.1: The phase portraits of flows $\Phi_1$ (left) and $\Phi_2$.

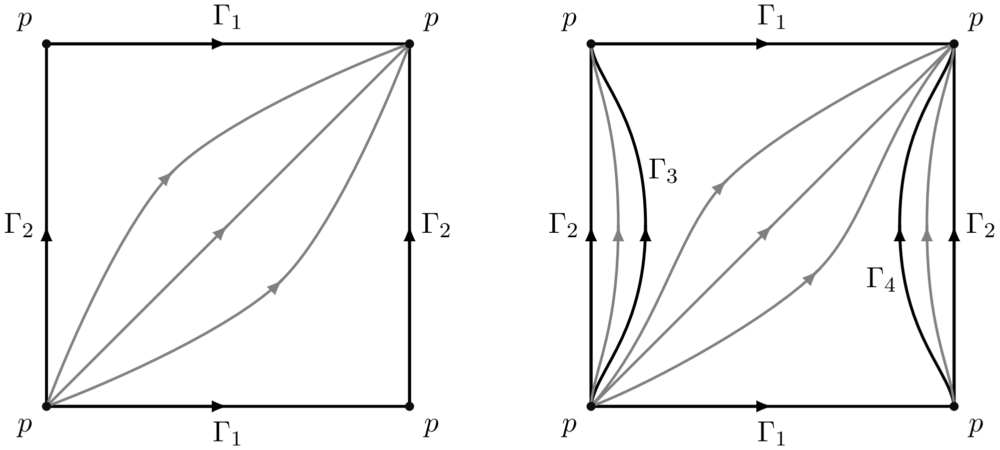


Figure 3.2: The phase portraits of flows $\Phi_3$ (left) and $\Phi_4$.

all orbits are homoclinic. As it happens, $p$ is the only separatrix for both flows. To show, say, that $\Gamma_1$ is ordinary (for $\Phi_4$), remove from $\mathbb{T}^2$ the closure of the strip delimited by $\Gamma_3$ and $\Gamma_4$ and containing $\Gamma_2$, to get a radial region containing $\Gamma_1$ with boundary $\Gamma_3 \cup \Gamma_4 \cup \{p\}$. In the case of $\Phi_3$, the radial region $\Omega' = \mathbb{T}^2 \setminus (\Gamma_2 \cup \{p\})$ cannot be used (there is just one regular orbit in its boundary), but we take off another orbit $\Gamma$ and use the strip $\Omega = \Omega' \setminus \Gamma$ instead. Once again, the separatrix configurations $\mathcal{S}_3^+ = \mathcal{S}_4^+ = \{p\} \cup \Gamma_1$ coincide, but $\Phi_3$ and $\Phi_4$ are not equivalent because $\Phi_3$ has three unstable orbits ($p$, $\Gamma_1$ and $\Gamma_2$) and $\Phi_4$ has four ($p$, $\Gamma_1$, $\Gamma_3$ and $\Gamma_4$).

Clearly, the problem with the previous examples is that the neighbouring regions we are using for ordinary orbits are, so to speak, too "big", and as a consequence the bounding orbits are not what they are "supposed" to be. A way to avoid this is not allowing parallel

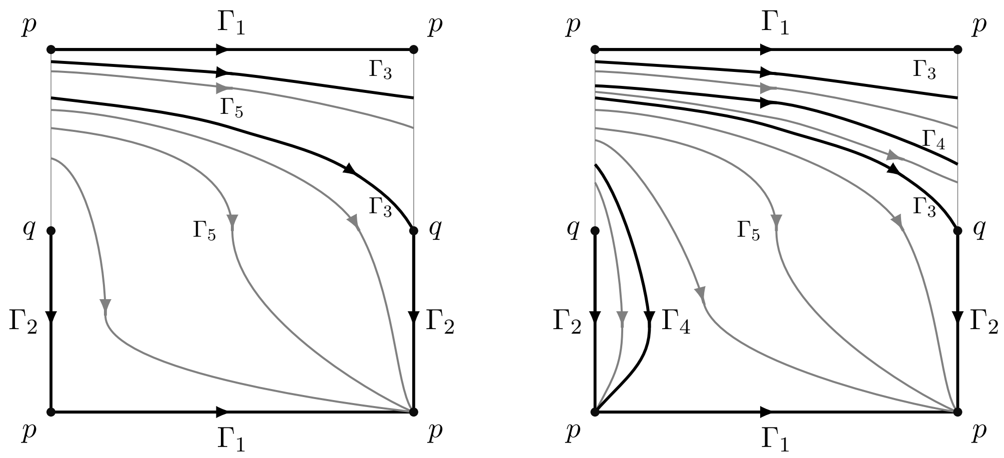


Figure 3.3: The phase portraits of flows $\Phi_5$ (left) and $\Phi_6$.

regions to be radial (trivially they cannot be toral either) in Definition 3.1. Moreover, we can force strips to be "strong". More precisely, we say that a strip $\Omega$ is *strong* if there are non-recurrent orbits $\Gamma_1, \Gamma_2$ such that $(\Omega', \Phi)$ is topologically equivalent to the restriction of the flow $\Phi_s$ to $\mathbb{R} \times [-1, 1]$, where $\Omega' = \Omega \cup \Gamma_1 \cup \Gamma_2$. We call $\Gamma_1$ and $\Gamma_2$ the *border orbits* of the strip $\Omega$, and say that a complete transversal to $\Omega$ is *strong* if it can be extended to an arc by adding one point from each border orbit. All orbits from a strip are non-recurrent: by requiring that the border orbits of a strong strip also are, we get rid of the annoying distinction between $\alpha_\Phi(p)$ and $\alpha'_\Phi(p)$ or $\omega_\Phi(p)$ and $\omega'_\Phi(p)$ (in the annular case, Definition 3.1(i) and (ii) are quite redundant, anyway).

Unexpectedly, Theorem 3.2 keeps failing even after redefining ordinary orbits as in the paragraph above, see Figure 3.3. In this torus example, common orbits to $\Phi_5$ and $\Phi_6$ are $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_5$ and the singular points $p$ and $q$. Observe that typical orbits of these flows have $\Gamma_1 \cup \{p\}$ as their $\alpha$-limit set and $\{p\}$ as their $\omega$-limit set. Checking that $\Phi_5$ and $\Phi_6$ are not equivalent, while having the same separatrix configurations $\mathcal{S}_5^+ = \mathcal{S}_6^+ = \{p, q\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5$, is as simple as in the previous examples.

The underlying problem here is that $\alpha$-limit and $\omega$-limit sets should be separately managed by Definition 3.1, but they are not. The orbit $\Gamma_4$ (for $\Phi_6$) is neighboured by strong strips, as close to it as required, whose boundaries consist of (as prescribed) the border orbits, the $\alpha$-limit set $\Gamma_1 \cup \{p\}$ and, *a fortiori*, the $\omega$-limit set $\{p\}$. Nevertheless, after removing a strong transversal from the strip, the boundary of the forward semi-strip contains, besides the border semiorbits, the strong transversal and the $\omega$-limit point $\{p\}$, the "spurious" orbit $\Gamma_1$.

## 3.3 A new statement of the theorem; an improvement

Taking all the above into consideration, we define:

**Definition 3.4.** We say that an orbit $\varphi(p)$ of $(S, \Phi)$ is *almost fine* if it is neighboured by an annular region or by a strong strip $\Omega$ with border orbits $\varphi(a)$ and $\varphi(b)$ such that:

(i) $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$ for any $q \in \Omega \cup \varphi(a) \cup \varphi(b)$;

(ii) $\operatorname{Bd} \Omega = \varphi(a) \cup \varphi(b) \cup \alpha(p) \cup \omega(p)$.

If $\varphi(p)$ satisfies the analogous conditions, replacing (ii) by

(ii') if $T$ is a strong transversal to $\Omega$ with endpoints $a$ and $b$, then $\operatorname{Bd} \Omega_T^- = T \cup \varphi(-\infty, a) \cup \varphi(-\infty, b) \cup \alpha(p)$ and $\operatorname{Bd} \Omega_T^+ = T \cup \varphi(a, \infty) \cup \varphi(b, \infty) \cup \omega(p)$,

then we say that $\varphi(p)$ is *fine*.

If an orbit is not fine, then it is called a *separator*.

Observe that the union set of all separators is closed as well, when the components of its complementary set will be called *standard regions*. Since all separatrices are separators, every standard region is contained in a canonical region. The notions of *separator configuration* and of *equivalence of separator configurations* are accordingly defined.

*Remark* 3.5. Typically, books and papers invoking the Markus-Neumann Theorem in the setting of analytic sphere flows (in particular, after carrying polynomial planar flows to the sphere via the Bendixson or the Poincaré projections), use an alternative definition of separatrix, see for instance [68, Section 3.11]. Here, under the additional assumption of finiteness of singular points, an orbit is called a "separatrix" if and only if it is either a singular point, a limit cycle, or an orbit lying in the boundary of an hyperbolic sector. Using the finite sectorial decomposition property for isolated (non-centers) singular points of analytic flows, noting that analyticity excludes the existence of one-sided isolated periodic orbits, and recalling some basic Poincaré-Bendixson theory, it is not difficult to show that this notion is, in fact, equivalent to that of separator (see also Proposition 3.6(b) or (c) below). Of course, as emphasized by our first counterexample (and contrarily to that stated in [68]) there may be orbits bounding hyperbolic sectors which are not separatrices in the Markus-Neumann sense.

Note, finally, that the previous discussion make no sense outside the sphere (just think of the irrational flow on the torus: here all orbits are separatrices).

Trivially, a fine orbit is almost fine. The converse is not true, as shown by the flow $\Phi_6$. Nevertheless, we have (recall the definition of essential point in Section 1.4.3):

**Proposition 3.6.** *Let $\varphi(p)$ be an almost fine orbit of $(S, \Phi)$. Assume that one of the following conditions holds:*

*(a) Both $\alpha(p)$ and $\omega(p)$ are finite (that is, empty or consisting of one point);*

*(b) $S$ has zero genus and the set of essential singular points of $\Phi$ is totally disconnected;*

*(c) $S = \mathbb{R}^2$ or $S = \mathbb{S}^2$.*

*Them $\varphi(p)$ is fine.*

*Proof.* In all three cases we must show that if $\Omega$ is a strong strip satisfying (i) and (ii) in Definition 3.4, then (ii') holds as well.

Assume that (a) holds. We just prove (the other equality is analogous) $\operatorname{Bd}\Omega_T^- = T \cup \varphi(-\infty, a) \cup \varphi(-\infty, b) \cup \alpha(p)$, when we can assume (otherwise the statement is trivial) $\alpha(p) = \{u\} \neq \omega(p)$. Then there is a small topological disk $D$, neighbouring $u$ (hence not intersecting $\omega(p)$), such that $\varphi(a)$ and $\varphi(b)$ intersect $D$ at respective semiorbits $\varphi(-, a')$, $\varphi(-, b')$; moreover, this can be done so that one of the arcs in $\operatorname{Bd} D$ joining $a'$ and $b'$ is the closure of a strong transversal $T'$ to $\Omega$. One of the regions into which $\varphi(-, a') \cup \{u\} \cup \varphi(-, b')$ decomposes $\operatorname{Int} D$ cannot intersect $\Omega$, and the other one, call it $\Omega'$, includes $\Omega_{T'}^-$. By the hypothesis on $\operatorname{Bd}\Omega$, and the fact that $\omega(p)$ does not intersect $D$, both $\Omega'$ and $\Omega_{T'}^-$ have the same boundary, hence (because they are connected) $\Omega' = \Omega_{T'}^-$. This implies the statement.

Now suppose that (b) holds. Because $S$ has zero genus, there is no loss of generality in assuming that it is a region in $\mathbb{S}^2$ (see Theorem 1.33). Observe that, since $\varphi(p)$ is almost fine, all non-essential singular points contained in $\alpha(p) \cup \omega(p)$ must be horizontal. If $\alpha(p) \cup \omega(p)$ contains a regular point or an horizontal singular point, then a standard Poincaré-Bendixson argument allows to find a semiorbit $\varphi(c, d)$ of $\varphi(p)$, and a transversal joining $c$ and $d$, whose union is a circle decomposing $\mathbb{S}^2$ into two regions, one including $\alpha(p)$, the other one including $\omega(p)$. By the compactness of $\{a, b\} \cup T$, there is $t_0$ such that $\Omega^-_{\Phi_{-t_0}(T)}$ is included in the first region, while $\Omega^+_{\Phi_{t_0}(T)}$ is included in the second one, which easily implies that $\varphi(p)$ is fine. In the case when all points from $\alpha(p)$ and $\omega(p)$ are singular and essential, total disconnectedness implies finiteness and (a) applies.

Finally, suppose that (c) is true. If suffices to consider the case $S = \mathbb{S}^2$, as then the case $S = \mathbb{R}^2$ follows by passing to its one-point compactification, which is precisely $\mathbb{S}^2$ (recall Section 1.4.3). Moreover, as in (b), we can additionally assume that all points from $\alpha(p) \cup \omega(p)$ are singular. Let $U$ be the component of $\mathbb{S}^2 \setminus \operatorname{Sing}(\Phi)$ including $\varphi(p)$. Next define the equivalence relation $\sim$ in $\mathbb{S}^2$ by $u \sim v$ if $u = v$ or there is a component $C$ of $\mathbb{S}^2 \setminus U$ such that $u, v \in C$. As explained in Section 1.1.4 (alternatively see [9, p. 481]), the quotient space $\mathbb{S}^2/\sim$ is homeomorphic to $\mathbb{S}^2$ and the flow $\Phi$ collapses, in the natural way, to a flow $\Phi_\sim$ on $\mathbb{S}^2/\sim$, whose set of singular points is totally disconnected. By

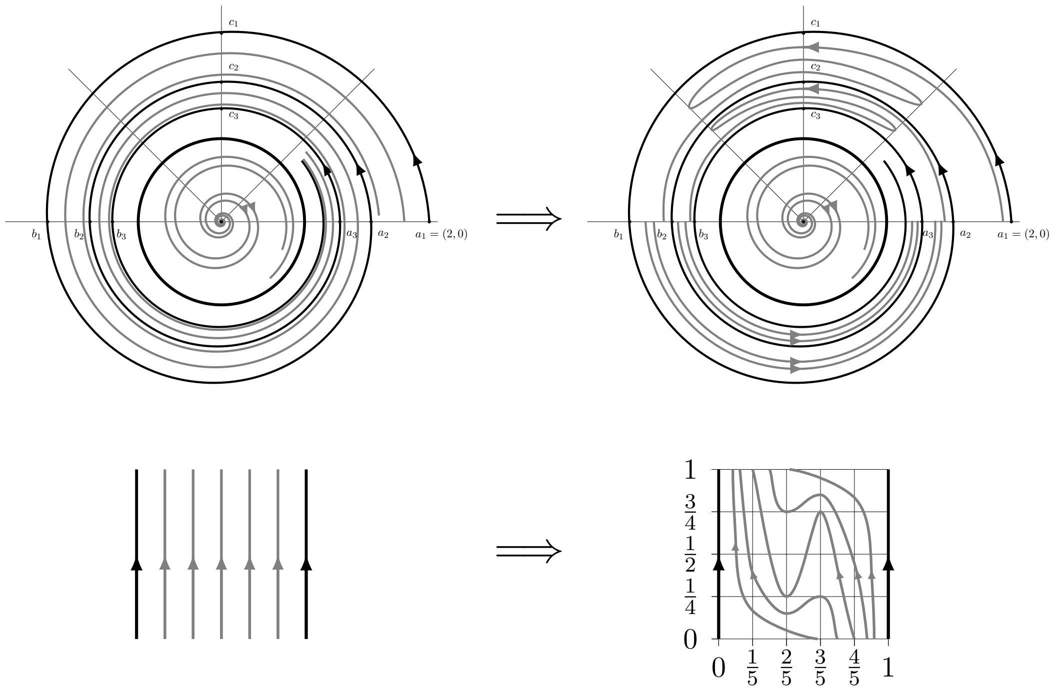


Figure 3.4: The phase portraits of flows $\Phi_7$ (left) and $\Phi_8$.

applying (b) to the collapsed flow, we deduce that the distances $d(\Phi_t(q), \alpha(p))$, $q \in \Omega_T^-$, tend uniformly to zero as $t \to -\infty$, and the same is true for $d(\Phi_t(q), \omega(p))$, $q \in \Omega_T^+$ and $t \to \infty$. Therefore, $\varphi(p)$ is fine. □

After extending the notion of separatrix as described in Definition 3.4, Theorem 3.2 works and, in fact, can be slightly improved, see Theorem B below. The improvement has to do with essential singular points. The left-hand flow $\Phi_7$ from Figure 3.4 is that associated (after deformation to clarify the picture outside the unit circle) with the vector field

$$f_7(x, y) = (1 - x^2 - y^2)\left(-(1 - x^2 - y^2)x - y, x - (1 - x^2 - y^2)y\right),$$

having the origin and the unit circle $\mathbb{S}^1$ as its set of singular points. Consecutive points of the semiorbit starting at $(2, 0)$ and intersecting the positive $x$-semiaxis (respectively, negative $x$-semiaxis, positive $y$-semiaxis) are denoted by $(a_n)_{n=1}^\infty$ (respectively, $(b_n)_{n=1}^\infty$, $(c_n)_{n=1}^\infty$). To construct $\Phi_8$ we modify, as indicated in the picture, the semiorbits from the regions enclosed by $\varphi_{\Phi_7}(a_n, b_n), \varphi_{\Phi_7}(a_{n+1}, b_{n+1})$, and the segments connecting $a_n$ and $a_{n+1}$ and $b_n$ and $b_{n+1}$. In the lower half-plane, and inside $\mathbb{S}^1$, the phase portrait does not change. Thus, for both flows, all regular orbits spiral towards $\mathbb{S}^1$ in positive time, and if $\Gamma_1$ denotes the orbit passing through $(2, 0)$ and $\Gamma_2$ is an orbit inside $\mathbb{S}^1$, then $\{\mathbf{0}\} \cup \mathbb{S}^1 \cup \Gamma_1 \cup \Gamma_2$ is a separator configuration for both $\Phi_7$ and $\Phi_8$.

Nevertheless, these flows are not equivalent. The key point is that, while all semi-lines starting from the origin are transversal (except at the unit circle) to $\Phi_7$, transversals connecting the points $(c_n)_n$ for $\Phi_8$ cannot be fully included in the octant $\{(x,y) : y > |x|\}$, hence their diameters are uniformly bounded, from below, by a positive number. Now assume that $h$ is a topological equivalence mapping the orbits of $\Phi_8$ onto those of $\Phi_7$. Then the points $h(c_n)$ converge to a point $d \in \mathbb{S}^1$, and there are transversals $T_n$ (for $\Phi_7$) connecting the points $h(c_n)$ and $h(c_{n+1})$ whose diameters tend to zero. Hence the diameters of the transversals $h^{-1}(T_n)$ (for $\Phi_8$) also tend to zero, which is impossible because they connect the points $c_n$ and $c_{n+1}$.

There is no contradiction with Theorem 3.2 here, because the set of singular points is not discrete. On the other hand, note that all singular points in the octant are essential for the flow $\Phi_8$, which is *really* the reason why Theorem 3.2 fails in this case:

**Theorem B.** *Let $S$ be a surface and suppose that $\Phi_1$ and $\Phi_2$ are flows on $S$ whose sets of essential singular points are discrete. Then $\Phi_1$ and $\Phi_2$ are topologically equivalent if and only if they have equivalent separator configurations.*

We outline the proof of Theorem B in the next section. Let us presently emphasize the usefulness of the improved condition on the singular points. On the one hand, recall that it implies, in the zero genus case, that all almost fine orbits are fine (Proposition 3.6(b)). On the other hand, we have:

**Proposition 3.7.** *If $(S,\Phi)$ is associated with an analytic vector field $X$, then either $X$ is identically zero or its set of essential singular points is discrete.*

*Proof.* If $\mathrm{Sing}(\Phi)$ has nonempty interior, the analyticity of $X$ and the connectedness of $S$ implies that $\mathrm{Sing}(\Phi) = S$ (see Proposition 1.7); we discard this trivial case in the rest of the proof.

First, for every $p \in S$, there exist a neighbourhood $U_p$ of $p$, an analytic map $\rho_p : U_p \to \mathbb{R}$ and an analytic vector field $Y_p$ on $U_p$, such that the restriction of $X$ to $U_p$ equals $\rho_p Y_p$ and the vector field $Y_p$ has no zeros in $U_p \setminus \{p\}$ (see Theorem 1.9). If $S_1$ denotes the set of singular points with the property that any $Y_p$ in such a decomposition vanishes at $p$, then $S_1$ is clearly closed and discrete.

Since $\mathrm{Sing}(\Phi)$ is the set of zeros of the analytic map $X_1^2 + X_2^2$, $X = (X_1, X_2)$, by virtue of Theorem A, $\mathrm{Sing}(\Phi)$ is locally, at each of its points $p$, a topological star with finitely many branches ("zero" branches meaning that the point is isolated in $\mathrm{Sing}(\Phi)$); moreover, the closure of any such branch $B$ can be parametrized via a bijective analytic map $\varphi : [0,1] \to B$ with $\varphi(0) = p$ (see Remark 2.1). Clearly, the set $S_2$ of points where $\mathrm{Sing}(\Phi)$ is not locally a 2-star (that is, there is not an arc neighbouring $p$ in $\mathrm{Sing}(\Phi)$) is also closed and discrete.

To finish the proof, it then suffices to show that if $S_3$ is the set of essential singular points not included in $S_1 \cup S_2$, then $S_3$ is closed and discrete as well. Let $p \in S_3$ and assume $U_p$ to be small enough so that it is a tubular neighbourhood of $p$ for the non-vanishing vector field $Y_p$. We can also assume that there is an analytic bijection $\lambda_p$ : $(-1,1) \to \mathrm{Sing}(\Phi) \cap U_p$. Since $p$ is not vertical, $\lambda_p'(0)$ must be parallel to $Y_p(p)$, that is, $T_p(s) = \lambda_{p,1}'(s)Y_{p,2}(\lambda_p(s)) - \lambda_{p,2}'(s)Y_{p,1}(\lambda_p(s))$ vanishes for $s = 0$, and since $p$ is not horizontal, $T_p(s)$ cannot be identically zero. Analyticity then implies that there is $\epsilon > 0$ such that $T_p(s)$ does not vanishes at $(-\epsilon, \epsilon) \setminus \{0\}$, that is, all singular points close enough to $p$ are vertical. In particular, $S_3$ is discrete.

To prove that $S_3$ is closed, it suffices to show that if $(p_n)_n$ is a sequence of pairwise distinct points of $S_3$, then it cannot converge. Assume the opposite and call $p$ its limit, when we can also assume that all points $p_n$ belong to the same branch $B$ of the star of singular points with centre $p$ and are included in the neigbourhood $U_p$. Find an analytic parametrization $\varphi : [0,1] \to B$ as previously explained, with $\varphi(t_n) = p_n$ and $t_n \to 0$, and realize that vectors $\varphi'(t_n)$ and $Y_p(\varphi(t_n))$ are parallel for all $n$. Hence, $\varphi'(t)$ and $Y_p(\varphi(t))$ are parallel for all $t \in [0,1]$, which is to say that all points $p_n$ are, in fact, horizontal. This contradiction finishes the proof. □

**Corollary 3.8.** *Let $S$ be a surface and suppose that $\Phi_1$ and $\Phi_2$ are flows on $S$ associated with analytic vector fields. Then $\Phi_1$ and $\Phi_2$ are equivalent if and only if they have equivalent separator configurations.*

## 3.4 Why the proof of Theorem 3.2 fails, and how to prove Theorem B

Roughly speaking, the proof of Theorem 3.2 by Markus and Neumann goes as follows. First of all, it is shown that each canonical region for a flow $(S, \Phi)$ is parallel. (The same reasoning still works, word by word, for standard regions; alternatively, notice that each invariant region in a parallel region is parallel as well.) Here observe that, by a simple connectedness argument, all orbits in a canonical (or a standard) region $\Omega$ share their $\alpha$-limit sets and their $\omega$-limit sets. Thus it make sense to write $\alpha(\Omega)$ and $\omega(\Omega)$, respectively, to denote them.

Next, under the hypotheses of Theorem 3.2 for $(S, \Phi_1)$ and $(S, \Phi_2)$, an easy simplification allows to assume that both separatrix configurations are equal, $\mathcal{S}^+ := \mathcal{S}_1^+ = \mathcal{S}_2^+$, hence the canonical regions of $(S, \Phi_1)$ and $(S, \Phi_2)$ are also equal and the topological equivalence $h : S \to S$ we are looking for should map each canonical region into itself. Note that the existence of a toral canonical region implies that $S = \mathbb{T}^2$; this trivial case can be discarded, for then both $(\mathbb{T}^2, \Phi_1)$ and $(\mathbb{T}^2, \Phi_2)$ are equivalent to the rational flow $(\mathbb{T}^2, \Phi_{ss})$.

Now the difficult part of the proof comes (Section 3 in [63] and Section 7 in [59]): starting from assuming that $h$ is the identity on $\mathcal{S}^+$, it must be homeomorphically extended to each canonical region $\Omega$ (mapping orbits from $(S, \Phi_1)$ into orbits from $(S, \Phi_2)$ and preserving the time directions). After explaining how this extension must be done, the authors first check the continuity from "inside" at the so-called accessible regular points from $\mathrm{Bd}\,\Omega$ (by *accessible* we mean that there is a lateral tubular region at the point which is included in $\Omega$), then deduce the continuity from "outside" and at the rest of regular points in $\mathcal{S}^+$, and finally prove the continuity at the (isolated) singular points. If fact, the argument equally works under the weaker hypothesis that the sets of essential singular points are discrete. Continuity at vertical singular points is guaranteed from the very beginning, because they are interior to $\mathcal{S}^+$; on the other hand, maximal curves of horizontal singular points can be dealt with exactly as if they were regular orbits.

Unfortunately, in their construction Markus and Neumann take for granted the following intuitively obvious (but, as shown by the counterexamples from the previous section, not necessarily true) fact: if a transversal to a canonical region ends at an accesible point from its boundary, then the transversal must be semi-complete. Using standard regions allows to override this difficulty:

**Proposition 3.9.** *Let $\Omega$ be a strip, annular or radial standard region. If $p \in \mathrm{Bd}\,\Omega$ is a regular or a horizontal singular point, and $L \subset \Omega$ is a transversal ending at $p$ (that is, there is an arc $A$ with endpoint $p$ such that $A' = A \setminus \{p\} \subset L$), then $L$ is semi-complete; more precisely, there is a complete transversal to $\Omega$ including $A'$.*

*Proof.* When $\Omega$ is annular, the result is clear.

Assume now that $\Omega$ is a strip and fix a topological equivalence $g$ between $(\Omega, \Phi)$ and $(\mathbb{R}^2, \Phi_s)$, when there is no loss of generality in assuming that $g$ preserves directions (that is, if $T = g^{-1}(\{0\} \times \mathbb{R})$, then $\Omega_T^+ = g^{-1}((0, \infty) \times \mathbb{R}))$ and $g^{-1}(0,0) = q$ is the other endpoint of the arc $A$.

Let $I = (c, d)$ $(-\infty \leq c < 0 < d \leq \infty)$ be the open interval and $\mu : I \to \mathbb{R}$ be the continuous map such that $g(L) = \{(\mu(s), s) : s \in I\}$ when, say, $\lim_{s \to d} h^{-1}((\mu(s), s)) = p$. We argue to a contradiction by assuming that $d < \infty$.

We claim that either $\lim_{s \to d} \mu(s) = \infty$ or $\lim_{s \to d} \mu(s) = -\infty$. Otherwise, there would be a sequence $s_n \to d$ with $\mu(s_n) \to r \in \mathbb{R}$, hence $g^{-1}((s_n, \mu(s_n))$ would converge both to $g^{-1}(d, r)$, a point in $\Omega$, and to $p$, which belongs to $\mathrm{Bd}\,\Omega$. This is impossible. We suppose, for instance, $\lim_{s \to d} \mu(s) = \infty$. Moveover, slightly modifying $g$ near $q$ if necessary, we can assume $\mu(s) > 0$ for all $s \in (0, d)$. Hence $T' = g^{-1}(\{0\} \times (0, \infty))$ does not intersect $A'$.

Let $v = h^{-1}(0, d)$. The orbit $\varphi(v)$ is fine, so there is a strong strip $U \subset \Omega$ neighbouring it, with its border orbits also included in $\Omega$, verifying Definition 3.4(ii'). Since the points in $\mathrm{Cl}(S_{T \cap U}^+)$ which are not in $\Omega$ belong to $\omega(\Omega) = \omega(v)$, and $p$ is one of such points because $g^{-1}(\mu(s), s)$ is included in $S$ if $s$ is close enough to $d$, we get $p \in \omega(\Omega)$.

Fix now a couple of lateral tubular regions $V$ and $W$ at $p$. We can assume that $V \subset \Omega$

and, moreover, $A \subset V$. Let $B$ be the corresponding lateral transversal at $p$ included in $W$. Since $p \in \omega(\Omega)$, all positive semiorbits $\varphi(z,+)$, $z \in T$, must intersect $B$ (in fact, infinitely many times). Let $q^*$ be the first point from $\varphi(q,+)$ in $B$, and denote by $B' \subset B$ the transversal with endpoints $p$ and $q^*$. Now let $T'_0$ (respectively, $T'_1$) be the set of points $z \in T'$ such that the first intersection point of $\varphi(z,+)$ with $A' \cup B'$ belongs to $A'$ (respectively, to $B'$). Both sets are disjoint and nonempty ($v \in T'_1$, and all points from $T' \cap V$, in particular those close enough to $q$, belong to $T'_0$), its union is the whole $T'$, and they are clearly open in $T'$ because the orbit $\varphi(q)$ does not intersects $T'$. This contradicts the connectedness of $T'$.

Finally, we assume that $\Omega$ is radial and reason again by way of contradiction, assuming that $A'$ does not intersect all orbits of $\Omega$ infinitely many times. It is clear that, without loss of generality, we can suppose that $A'$ does not meet every single orbit in $\Omega$; with more detail, there is no restriction in assuming that there exist some $z \in \Omega$ and some strip neighbourhood of $\varphi_\Phi(z)$, $U$, such that $U \cap A' = \emptyset$.

Now consider a new flow $\Phi'$ having exactly the same orbits as $\Phi$ in $S \setminus \varphi_\Phi(z)$ and having $z$ as its only singular point. Then $\varphi_\Phi(z)$, when seen as a subset of $(S, \Phi')$, consists of three separators for $\Phi'$: the singular point $z$ and two regular orbits given by the components of $\varphi_\Phi(z) \setminus \{z\}$. Moreover, $\Omega' = \Omega \setminus \varphi_\Phi(z)$ is a strip and, clearly, a standard region for $\Phi'$. The previous argument implies that $A'$ is semi-complete for $\Phi'$, which is impossible because it does not intersect $U$. $\square$

With the help of Proposition 3.9, Theorem B can be proved, without further changes, as explained above.

# Chapter 4

# Topological classification of unstable global attractors of polynomial planar vector fields

In the present chapter we present the results collected in [28]: we classify polynomial global attraction up to topological equivalence. Indeed we work in the much more general setting of planar flows with finitely many separators (or equivalently, see Remark 4.21, those having the finite sectorial decomposition property at $\mathbf{0}$, or those having finitely many unstable orbits), when their separator configurations are also finite. To begin with, there is a dichotomy: global attraction is trivial if and only if $\mathbf{0}$ is positively stable, that is, there are no regular homoclinic orbits (Proposition 4.14). Hence we concentrate in what follows in the "non-positively stable" case, when at least (as implied by Proposition 4.14) one heteroclinic separator must exist. We rely on Theorem B: two flows are equivalent if and only if there is a plane homeomorphism preserving the orbits and time directions of their separator configurations. As it turns out, a weaker so-called *compatibility* condition (just assuming preservation of orbits, see Subsection 4.1.2) suffices, provided that at least one heteroclinic separator is preserved as well. Moreover, after fixing an orientation in $\mathbb{R}^2$ (counterclockwise or clockwise) and a heteroclinic separator, and using the separator configuration combinatorial structure, there is a canonical way to associate a so-called feasible set (a finite vectorial set as described in Definition 4.18) to the flow, and this labelling characterizes equivalence: topologically equivalent flows have the same canonical feasible set. We emphasize that although the separator configuration is not uniquely defined, no ambiguity arises because the corresponding canonical feasible sets are the same (this follows from Lemma 4.13).

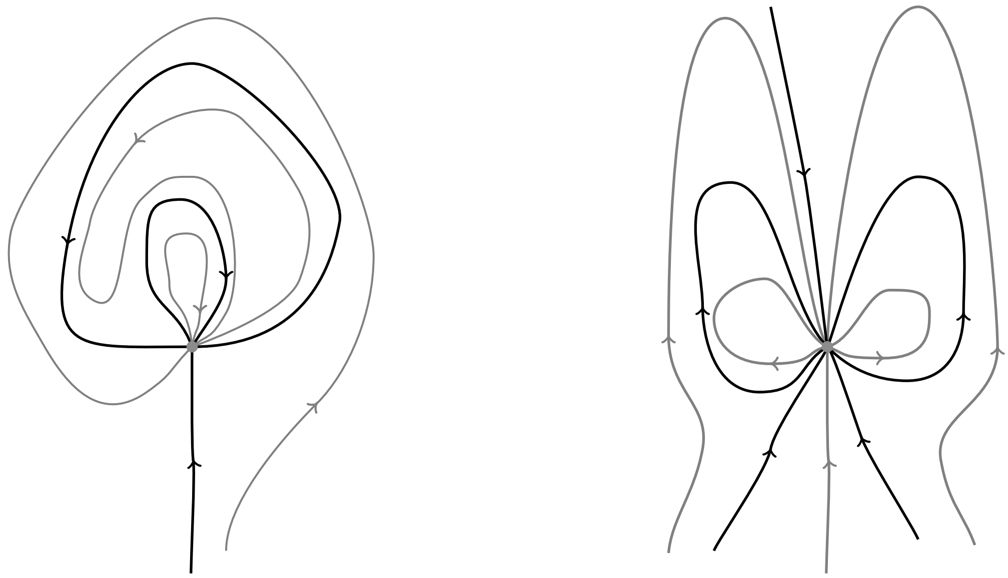

Figure 4.1: Two non-equivalent phase portraits with the same sectorial decomposition (elliptic-elliptic-hyperbolic-attracting-hyperbolic in counterclockwise sense) at the origin.

Our first theorem summarizes these results.

**Theorem C.** *Assume that* $\mathbf{0}$ *is a global attractor, non-positively stable, for two plane flows* $\Phi$ *and* $\Phi'$*, both having finitely many separators, and let* $\mathcal{X}$ *and* $\mathcal{X}'$ *denote their separator configurations. Then the following statements are equivalent:*

*(i)* $\Phi$ *and* $\Phi'$ *are topologically equivalent.*

*(ii)* $\mathcal{X}$ *and* $\mathcal{X}'$ *are compatible and the compatibility bijection* $\xi : \mathcal{X} \to \mathcal{X}'$ *maps some heteroclinic separators of* $\Phi$ *to a heteroclinic separators of* $\Phi'$*.*

*(iii) There are respective orientations* $\Theta, \Theta'$ *in* $\mathbb{R}^2$ *and heteroclinic separators* $\Sigma, \Sigma'$ *such that the associated canonical feasible sets are the same.*

We remark that sharing (up to homeomorphisms) the same finite sectorial decomposition is a necessary but not sufficient condition for two such flows being topologically equivalent, see Figure 4.1. Likewise, compatibility alone is not enough to guarantee topological equivalence, see Figure 4.2.

Although the lemmas in Section 4.2 do not require finiteness of separators, no attempt has been done to find a more general version of Theorem C disposing of this restriction. Anyway, we are mainly interested in polynomial (local) flows, that is, those associated with polynomial vector fields, hence finiteness of separators is guaranteed (Remarks 1.57 and 4.2). Our next result, together with Theorem C, implies that if a flow has a globally attracting singular point, then it is topologically equivalent to a polynomial flow.

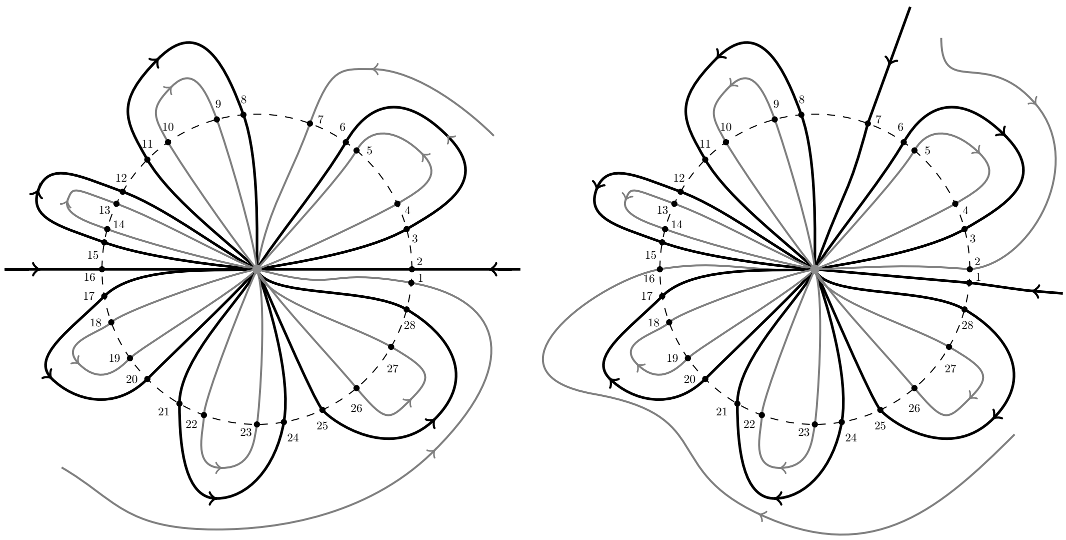


Figure 4.2: Two non-equivalent phase portraits with compatible separator configurations (numbering indicating the compatibility bijection).

**Theorem D.** *Let $L$ be a feasible set. Then there are a polynomial flow $\Phi$ (having $\mathbf{0}$ as a non-positively stable global attractor) and a heteroclinic separator $\Sigma$ of $\Phi$ such that $L$ is the canonical feasible set associated with $\Phi$, the counterclockwise orientation in $\mathbb{R}^2$ and $\Sigma$.*

Our proof of Theorem D depends heavily on the paper [74], where sufficient conditions are given allowing the flow associated with a $C^1$-vector field to be equivalent to a polynomial flow. In a sense this is not fully satisfying, because the arguments in [74] are essentially non-constructive. In fact, to the best of our knowledge, the literature provides no explicit examples of polynomial flows having a nontrivial globally attracting singular point. For this reason we finally prove:

**Theorem E.** *The origin is both a global attractor and an* elliptic saddle *for the system*

$$\begin{cases} x' = -((1+x^2)y + x^3)^5, \\ y' = y^2(y^2 + x^3), \end{cases} \tag{4.1}$$

*that is, the origin possesses a neighbourhood decomposed as exactly one elliptic sector and one hyperbolic sector.*

## 4.1 Preliminary notions

While polynomial planar vector fields are the primary interest of this chapter, and their associated flows are usually just local, there is a way to get rid of this restriction. Indeed, for any local flow $\Phi$ on $\mathbb{R}^2$ there exists a (global) flow of class $C^\infty$ on $\mathbb{R}^2_\infty$ which has $\infty$ as a singular point and whose restriction to $\mathbb{R}^2$ is topologically equivalent to $\Phi$ (see Subsection 1.4.3). To simplify the notation we will call $\Phi$, rather than $\Phi_\infty$, this extended flow, hoping that this will not lead to confusion. If $\Phi$ is associated with a polynomial planar vector field, then we also call it (and its extension) *polynomial*, although of course this map is not "polynomial" in the usual sense.

Conversely, there is a natural way to transport polynomial vector fields from $\mathbb{S}^2$ to $\mathbb{R}^2$. Namely, if $f : \mathbb{S}^2 \to \mathbb{R}^3$ is a polynomial vector field, tangent to $\mathbb{S}^2$ and vanishing at the north pole $p_N = (0,0,1)$ of $\mathbb{S}^2$, say $f(u,v,w) = (P(u,v,w), Q(u,v,w), R(u,v,w))$, then we can carry it, via the stereographic projection, to the plane vector field

$$g(x,y) = (1-w)^{-1}(P(u,v,w) + R(u,v,w)x, Q(u,v,w) + R(u,v,w)y)$$

with $u = 2x/(1+x^2+y^2)$, $v = 2y/(1+x^2+y^2)$, $w = (x^2+y^2-1)/(1+x^2+y^2)$, and after multiplying $g$ by an appropriate power of $1+x^2+y^2$ we obtain a polynomial vector field whose associated (polynomial) flow is topologically equivalent to the flow induced by $f$ on $\mathbb{S}^2 \setminus \{p_N\}$.

### 4.1.1 On special flows and regions

The standing assumption in this section is that $\mathbf{0}$ is a globally attracting singular point for the flows $\Phi$ on $\mathbb{R}^2$ we deal with, that is, $\omega(z) = \{\mathbf{0}\}$ for any $z \in \mathbb{R}^2$. This is closely related to the notions of heteroclinicity and homoclinicity. In this context, an orbit $\varphi(z)$ of $\Phi$ is *homoclinic* (respectively, *heteroclinic*) if $\omega(z) = \{\mathbf{0}\}$ and $\alpha(z) = \{\mathbf{0}\}$ (respectively, $\alpha(z) = \emptyset$ —that is, $\alpha(z) = \{\infty\}$ when using the extended flow to $\mathbb{R}^2_\infty$). Of course, the singular point $\mathbf{0}$ is trivially homoclinic. If $\Gamma$ is homoclinic, then we denote by $E(\Gamma)$ the disk enclosed by the circle $\Gamma \cup \{\mathbf{0}\}$ (or just the singleton $\{\mathbf{0}\}$ in the case $\Gamma = \{\mathbf{0}\}$). When all orbits of a flow are heteroclinic or homoclinic, then it trivially has $\mathbf{0}$ as a global attractor. The converse is true as well (Lemma 4.5).

In Chapter 3, we introduced the notions of parallel regions for a flow. Notice that in the context of this chapter, only the strip and the radial regions are compatible with the properties of the flows we are dealing with ($\mathbf{0}$ is a global attractor).

If all orbits of a strip $\Omega \subset \mathbb{R}^2$ are heteroclinic (respectively, homoclinic), then we call $\Omega$ *heteroclinic* (respectively, *homoclinic*) as well. When $\Omega$ is a strong heteroclinic (or homoclinic) strip, with $\Gamma_1$ and $\Gamma_2$ as border orbits (when notice that, due to Lemma 4.6, they both are also heteroclinic or homoclinic, respectively), and $\mathrm{Cl}(\Omega) = \Omega \cup \Gamma_1 \cup \Gamma_2 \cup \{\mathbf{0}\}$, then we say that $\Omega$ is *solid*.

If $Q$ is a transversal circle (respectively, open arc) with the property that, for every $z \in Q$, $\varphi(z)$ intersects $Q$ exactly at $z$, then $\Omega = \bigcup_{z \in Q} \varphi(z)$ is a radial (respectively strip) region. To construct the corresponding homeomorphism $g : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \Omega$ (respectively,

$g : \mathbb{R}^2 \to \Omega$) just fix a homeomorphism $f : \mathbb{S}^1 \to Q$ (respectively, $f : \mathbb{R} \to Q$) and write $g(e^{-t+\mathbf{i}\theta}) = \Phi(t, f(e^{\mathbf{i}\theta}))$ (respectively $g((t,\theta)) = \Phi(t, f(\theta))$). Conversely, if $\Omega \subset \mathbb{R}^2$ is radial (respectively, a strip) then there is a circle (respectively, an open arc) $Q \subset \Omega$, transversal to $\Phi$, having exactly one common point with every orbit in $\Omega$. We call any such set $Q$ a *complete transversal* to $\Omega$ (notice that this notion is not in contradiction with the homonym notion introduced in Chapter 3 — in the strip case, we are given exactly the same concept, while in the radial case, we are here defining complete transversal circles and in Chapter 3 we worked with complete transversal open arcs). Also, recal that if $\Omega$ is a strong strip, then more is true: there is a transversal arc $T$ having exactly one common point with every orbit in $\Omega$ and every regular orbit in $\mathrm{Bd}(\Omega)$ (what we call, a strong transversal to $\Omega$).

*Remark* 4.1. If $\Omega$ is radial, and the circle $C$ is a complete transversal to $\Omega$, then it must enclose $\mathbf{0}$. Hence all heteroclinic orbits intersect $C$, that is, $\Omega$ is the union set of all heteroclinic orbits of $\Phi$; in other words, $\Phi$ admits one radial region at most (later we will see, Proposition 4.14, that such a region does exist). Moreover, the circles $\Phi_t(C)$ tend uniformly to $\infty$ as $t \to -\infty$. In fact, if $D_t \subset \mathbb{R}^2_\infty$ is the disk containing $\infty$ and having $\Phi_t(C)$ as its boundary, then $D_t = \{\Phi_s(u) : u \in C, s \leq t\} \cup \{\infty\}$. Since these disks intersect exactly at $\infty$, we get $\mathrm{diam}(D_t) \to 0$ as $t \to -\infty$, and the uniform convergence to $\infty$ follows. As a corollary, all heteroclinic orbits are negatively stable.

Similarly, if $\Omega$ is a solid strip and $T$ is a strong transversal to $\Omega$, then $\Phi_t(T)$ tends uniformly to $\mathbf{0}$ as $t \to \infty$, and tends uniformly to $\mathbf{0}$ as $t \to -\infty$ in the homoclinic case, and to $\infty$ in the heteroclinic case. In particular, all orbits of a solid strip are stable, and if it is heteroclinic (respectively, homoclinic), then the flow induced by $f_2$ on $\mathbb{R} \times [0,\infty)$ (respectively, by $f_4$ on the union set of $\mathbf{0}$ and all orbits intersecting the diagonal arc $\{(x,x) : 1/4 \leq x \leq 1/2\}$) is topologically equivalent to the restriction of $\Phi$ to $\mathrm{Cl}\,\Omega$ (recall the definitions of $f_2$ and $f_4$ given in Subsection 1.4.3).

In this context, the notion of separator (see Definition 3.4) can be reformulated as follows: an orbit is a separator of $\Phi$ if it is contained in no solid strip (c.f. Proposition 3.6).

*Remark* 4.2. As indicated in Remark 4.1, any unstable orbit must be a separator. If $\Phi$ has the finite sectorial decomposition property at $\mathbf{0}$, then $\Gamma$ is a separator if and only if it is either the singular point, or includes a semiorbit limiting a hyperbolic sector. In particular, $\Phi$ has finitely many separators and $\Gamma$ is a separator if and only if it is unstable.

The first statement in the next remark is a particular case of [59, Theorems 5.2 and 7.1], see also [63] and Chapter 3:

*Remark* 4.3. Notice that in the context of this chapter (the global attraction is incompatible with the existence of annular or toral parallel regions), any standard region of $\Phi$ must be either radial or a strip. On the other hand, it is clear that a strip (even a strong strip) needs not be either heteroclinic or homoclinic. Nevertheless, if a standard region is a strip, then it must be either heteroclinic or homoclinic (because, in this case, the set of its heteroclinic orbits and the set of its homoclinic orbits are both open; hence, by connectedness, one of them must be empty).

As particular case of Theorem B we have:

**Theorem 4.4.** *Assume that $\mathbf{0}$ is a global attractor for two flows $\Phi$ and $\Phi'$ and let $\mathcal{X}$ and $\mathcal{X}'$ denote some separator configurations for $\Phi$ and $\Phi'$. Then $\Phi$ and $\Phi'$ are topologically equivalent if and only if there is a homeomorphism from the plane onto itself mapping the orbits of $\mathcal{X}$ onto the orbits of $\mathcal{X}'$ and preserving the time directions.*

### 4.1.2 On orientations and the extension of homeomorphisms

Taking advantage of the results exposed in Subsection 1.1.4 (the notions and notation introduced there will be used here), we present an *ad hoc* way to relate sets of orbits of $\Phi$ and to use that relation to produce homeomorphism from the plane onto itself.

Let $C$ be a circle around $\mathbf{0}$. If $\Gamma$ is heteroclinic, we call the last point of $\Gamma$ in $C$ (that is, the point $q \in \Gamma \cap C$ such that $\Phi_q(t) \notin C$ for any $t > 0$) the *$\omega$-point* of $\Gamma$ in $C$. Likewise, if $\Gamma$ is regular and homoclinic and $C$ is small enough so that there are points of $\Gamma$ not enclosed by $C$, then we call the first and last points of $\Gamma$ in $C$ (that is, the points $p, q \in \Gamma \cap C$ such that $\Phi_p(t) \notin C$ for any $t < 0$ and $\Phi_q(t) \notin C$ for any $t > 0$) the *$\alpha$-point* and the *$\omega$-point* of $\Gamma$ in $C$, respectively.

If $\mathcal{P}$ is a finite family of orbits of $\Phi$, and $C$ is a circle around $\mathbf{0}$ small enough, then we denote by $\Delta_\Phi(\mathcal{P}, C)$ the set of all $\alpha$- and $\omega$-points in $C$ from the orbits in $\mathcal{P}$ and call it the *configuration* of $\mathcal{P}$ in $C$. Note that the possibility that the singular point belongs to $\mathcal{P}$ is not excluded, when of course it adds no points to $\Delta_\Phi(\mathcal{P}, C)$. Also, observe that all configurations of $\mathcal{P}$ are essentially the same, that is, if $C$ and $C'$ are small circles around $\mathbf{0}$, then there is an orientation preserving homeomorphism $h : C \to C'$ mapping the $\alpha$- and $\omega$-points in $C$ of every orbit $\Gamma \in \mathcal{P}$ to the $\alpha$- and $\omega$-points in $C'$ of that same orbit $\Gamma$.

If $\Gamma$ is homoclinic, then we say that it is *positive* (respectively, *negative*) when, after taking $\Gamma' \subset \operatorname{Int} E(\Gamma)$ and a small circle $C$ around $\mathbf{0}$, the $\alpha$- and $\omega$-points $p, q$ of $\Gamma$ in $C$, and the $\omega$-point $q'$ of $\Gamma'$ in $C$, we get that $(p, q', q)$ is positive (respectively, negative). In simpler words, $\Gamma$ is positive (negative) when the flow induces the counterclockwise (clockwise) orientation on $\Gamma \cup \{\mathbf{0}\}$.

Assume that $\mathcal{P}$ and $\mathcal{P}'$ are finite families of orbits of, respectively, $\Phi$ and $\Phi'$ (we also assume that both of them contain the globally attracting singular point $\mathbf{0}$ and at least one heteroclinic and one homoclinic orbit). Let $P$ and $P'$ be the union sets of these orbits and note that these sets are nice. Then, as it is simple to check, a condition characterizing the $\mathbb{R}^2$-compatibility of $P$ and $P'$ (when we accordingly say that $\mathcal{P}$ and $\mathcal{P}'$ are *compatible*) is the existence of a *compatibility bijection*. By this we mean a bijection $\xi : \mathcal{P} \to \mathcal{P}'$ for which there is a homeomorphism $\mu : C \to C'$, with $C$ and $C'$ small circles around $\mathbf{0}$, mapping $\Delta_\Phi(\mathcal{P}, C)$ onto $\Delta_{\Phi'}(\mathcal{P}', C')$, so that $\mu(C \cap \Gamma) = C' \cap \xi(\Gamma)$ for any $\Gamma \in \mathcal{P}$. In this case we say that $\mu$ *preserves orbits for* $\xi$.

If, additionally, $\mu$ maps $\omega$-points onto $\omega$-points (when we say that $\mu$ *preserves directions for* $\xi$), then the corresponding plane homeomorphism preserves the time directions on $\mathcal{P}$ and $\mathcal{P}'$. If, moreover, these families are the separator configurations of $\Phi$ and $\Phi'$, Theorem 4.4 implies that the flows are topologically equivalent.

## 4.2 General results on global attraction

Recall that we assume that $\mathbf{0}$ is a global attractor for $\Phi$.

**Lemma 4.5.** *All orbits of* $\Phi$ *are either homoclinic or heteroclinic.*

*Proof.* If the statement of the lemma is not true, then there is some point $z \in \mathbb{R}^2$ such that $\alpha(z)$ contains a regular point $u$. Let $T$ be a transversal to $u$. According to some well-known Poincaré-Bendixson theory, we can find $p, q \in \varphi(z) \cap T$ so that $\varphi(p, q) \cup S$ (where $S$ is the arc in $T$ whose endpoints are $p$ and $q$) is a circle enclosing a disk $D$ in $\mathbb{R}^2_\infty$ which contains $\varphi(-, p)$, and hence $\alpha(z)$, and intersects $\varphi(q, +)$ just at $q$. This is impossible: on the one hand, $\mathbf{0}$ cannot belong to $D$, because it is the $\omega$-limit set of $\varphi(q)$; on the other hand, $u \in \alpha(z)$ implies $\omega(u) \subset \alpha(z)$, so $\mathbf{0}$ does belong to $D$. $\square$

**Lemma 4.6.** *The union set of all homoclinic orbits of* $\Phi$ *is bounded.*

*Proof.* Assume the opposite to find a family of homoclinic orbits $\{\varphi(z_n)\}_{n=1}^\infty$ with $z_n \to \infty$ as $n \to \infty$ and fix a circle $C$ around $\mathbf{0}$. Using the continuity of the (extended) flow $\Phi$ at $\infty$, there is no loss of generality in assuming that the semiorbits $\Phi_{z_n}([-n, 0])$ do not intersect the region $O$ encircled by $C$. Next, find the numbers $a_n \leq -n$, closest to $-n$, such that the points $\Phi_{z_n}(a_n)$ belong to $C$ (using that the orbits $\varphi(z_n)$ are homoclinic) and assume, again without loss of generality, that the points $u_n = \Phi_{z_n}(a_n)$ converge to $u$. Since $\Phi_{u_n}(t) \in \mathbb{R}^2 \setminus O$ for any $t \in [0, n]$, the continuity of the flow implies that $\varphi(u, +)$ does not intersect $O$, contradicting that $\mathbf{0}$ is a global attractor. $\square$

Let $\mathcal{H}$ denote the family of homoclinic orbits of $\Phi$. We introduce a partial order in $\mathcal{H}$ by writing $\Gamma \preceq \Sigma$ if $\Gamma \subset E(\Sigma)$, when $\Gamma \prec \Sigma$ means of course $\Gamma \preceq \Sigma$ with $\Gamma \neq \Sigma$. We say that $\Gamma \in \mathcal{H}$ is *maximal* if there is no $\Sigma \in \mathcal{H}$ such that $\Gamma \prec \Sigma$. If $\Gamma, \Sigma \in \mathcal{H}$ and neither $\Gamma \preceq \Sigma$ nor $\Sigma \preceq \Gamma$ is true, then we say that $\Gamma$ and $\Sigma$ are *incomparable.* Realize that a family of pairwise incomparable orbits must be countable. Moreover, we have:

**Lemma 4.7.** *If the orbits* $\{\Gamma_n\}_{n=1}^\infty$ *are pairwise incomparable, then* $\operatorname{diam}(\Gamma_n) \to 0$ *as* $n \to \infty$.

*Proof.* Suppose the contrary to get a point $u \neq \mathbf{0}$ at which these orbits accumulate. Let $T$ be a transversal to $u$ and find points $u_{n_k} \in \Gamma_{n_k} \cap T$, $k = 1, 2, 3$, with, say, $u_{n_2}$ lying between $u_{n_1}$ and $u_{n_3}$ in $T$. Then $u_{n_1}$ and $u_{n_3}$ belong to different regions in $\mathbb{R}^2 \setminus (\Gamma_{n_2} \cup \{\mathbf{0}\})$: we are using here that any homoclinic orbit can intersect a transversal at one point at most. Thus, either $\Gamma_{n_1} \prec \Gamma_{n_2}$ or $\Gamma_{n_3} \prec \Gamma_{n_2}$, contradicting the hypothesis. □

**Lemma 4.8.** *Let $\Omega \subsetneq \mathbb{R}^2$ be a region invariant for $\Phi$.*

*(i) If $\Omega$ is bounded, then $\operatorname{Bd}\Omega$ is the union set of a homoclinic orbit $\Sigma$, a (possibly empty) family $\mathcal{G}$ of pairwise incomparable homoclinic orbits satisfying $\Gamma \prec \Sigma$ for every $\Gamma \in \mathcal{G}$, and the singular point.*

*(ii) If $\Omega$ is unbounded, then its boundary is the union set of at most two heteroclinic orbits, a (possibly empty) family of pairwise incomparable homoclinic orbits, and the singular point.*

*Proof.* Since $\Omega$ in invariant, $\operatorname{Bd}\Omega$ is invariant as well, and the statement (ii) follows easily from the connectedness of $\Omega$. To prove (i), assume that the boundary of the bounded region $\Omega$ is not as described and realize that then we must have $\operatorname{Bd}\Omega = \{\mathbf{0}\} \cup \bigcup_n \Gamma_n$ for a family $\{\Gamma_n\}_n$ (having at least two elements) of pairwise incomparable homoclinic orbits. Lemma 4.7 implies that $O = \mathbb{R}^2 \setminus \bigcup_n E(\Gamma_n)$ is a region including $\Omega$ with the same boundary as $\Omega$. Hence $\Omega = O$, contradicting that $\Omega$ is bounded. □

**Lemma 4.9.** *Let $\Gamma \in \mathcal{H}$. Then there is $\Sigma \in \mathcal{H}$, maximal for "$\prec$", such that $\Gamma \preceq \Sigma$.*

*Proof.* If $\Gamma$ is not maximal itself, then the Jordan curve theorem implies that the nonempty family $\mathcal{F} = \{\Gamma' \in \mathcal{H} : \Gamma \preceq \Gamma'\}$ is a totally ordered subset of $\mathcal{H}$; accordingly, it is enough to show that $\mathcal{F}$ has a maximal element for $\preceq$. Say $\mathcal{F} = \{\Gamma_i\}_i$. Then, because of the total ordering, $\Omega = \bigcup_i \operatorname{Int} E(\Gamma_i)$ is a region invariant for $\Phi$, and because of Lemma 4.6, $\Omega$ is bounded. As a result, we can apply Lemma 4.8(i) to obtain the corresponding homoclinic boundary orbit $\Sigma$. Then, clearly, $\Sigma$ is the maximal element of $\mathcal{F}$. □

*Remark* 4.10. Note that all maximal homoclinic orbits of $\Phi$ are separators.

**Lemma 4.11.** *Let $z$ be a regular point. Then there is a transversal $T$ to $z$ such that, for every $u \in T$, $\varphi(u)$ intersects $T$ exactly at $u$.*

*Proof.* Fix an arc $Q$ transversal to $z$. Note that no orbit can intersect $Q$ infinitely many times. Also, if some orbit intersects $Q$ at consecutive times $t < s$ and corresponding points $u$ and $v$, then no orbit can intersect the open arc in $Q$ with endpoints $u$ and $v$ more than once. Using these two facts it is easy to construct a transversal $T \subset Q$ to $z$ with endpoints $p$ and $q$ such that the orbits $\varphi(p)$ and $\varphi(q)$ intersect $T$ at exactly $p$ and $q$. This is the transversal we are looking for, because if an orbit $\Gamma$ consecutively intersects $T$ at points $u$

and $v$, and $D$ is the disk in $\mathbb{R}^2_\infty$ enclosed by $\varphi(u,v)$ and the arc in $T$ with endpoints $u$ and $v$ such that $\mathbf{0} \in D$, then either $\varphi(p)$ or $\varphi(q)$ does not intersect $D$, a contradiction. $\square$

*Remark* 4.12. If the boundary of a heteroclinic strip consists of the singular point and two heteroclinic orbits, then it is solid (the corresponding strong transversal can be found with the help of Lemma 4.11). If we replace "heteroclinic" by "homoclinic", this needs not happen unless we additionally assume that the ordering "$\prec$" totally orders the orbits of the strip.

**Lemma 4.13.** *If $\Omega$ is a standard region and $\Gamma, \Gamma'$ are distinct orbits in $\Omega$, then there is a solid strip $S \subset \Omega$ such that $\operatorname{Bd} S = \Gamma \cup \Gamma' \cup \{\mathbf{0}\}$.*

*Proof.* Let $Q$ be a complete transversal to $\Omega$ and let $A \subset Q$ be an arc with endpoints belonging to $\Gamma$ and $\Gamma'$. Since $\Omega$ includes no separators, for any point $z \in Q$ there is a solid strip in $\Omega$, containing $z$, whose closure intersects $Q$ at a small arc in $Q$ (this small arc thus being a strong transversal to the strip). Taking this into account, and applying a simple compactness argument to $A$, the lemma follows. $\square$

Recall that $\Phi$ admits one radial region at most, that consisting of all heteroclinic orbits of $\Phi$ (Remark 4.1). Indeed, such is the case:

**Proposition 4.14.** *Let $R$ be the union set of all heteroclinic orbits of $\Phi$. Then it is radial. Moreover:*

*(i) If $R = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, that is, all regular orbits of $\Phi$ are heteroclinic, then $\Phi$ is topologically equivalent to the associated flow with $f_2(x,y) = (-x,-y)$ in $\mathbb{R}^2$ (hence $\mathbf{0}$ is positively stable and it is the only separator of $\Phi$).*

*(ii) If $R \neq \mathbb{R}^2 \setminus \{\mathbf{0}\}$, then $R$ includes a separator of $\Phi$.*

*Proof.* First we assume $R = \mathbb{R}^2 \setminus \{\mathbf{0}\}$. To prove that $R$ is radial and (i) holds, it suffices to show that $\mathbf{0}$ is the only separator of $\Phi$ (Remark 4.3 and Theorem 4.4). Take $z \in R$ and let $T \subset R$ be an arc transversal to $z$ with the property that the orbits of all its points intersect $T$ exactly once (Lemma 4.11). Let $p$ and $q$ be the endpoints of $T$ and let $D$ be the disk in $\mathbb{R}^2_\infty$ enclosed by $\varphi(p)$, $\varphi(q)$, $\mathbf{0}$ and $\infty$ and including $T$. If $u \in \operatorname{Int} D$, then $\varphi(u)$ intersect $T$ (because it is heteroclinic). Therefore, $\operatorname{Int} D$ is a heteroclinic solid strip; in particular, $\varphi(z)$ is not a separator.

Assume now $R \neq \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Applying Lemma 1.4 to the union set $K = \mathbb{R}^2 \setminus R$ of all sets $E(\Gamma)$ with $\Gamma$ maximal for "$\prec$" (recall also Lemmas 4.7 y 4.9), and using (i), we can construct a topological equivalence between the restriction of $\Phi$ to $R$ and the restriction (to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$) of the associated flow with $f_2$. In particular, $R$ is radial.

To prove the last statement of the proposition, assume that $R$ includes no separators (hence it is a standard region by Remark 4.3), fix a complete transversal circle $C$ to $R$ and apply Lemma 4.13 (recall also Remark 4.1) to conclude the uniform convergence of $\Phi_t(C)$ to $\mathbf{0}$ and $\infty$ as $t \to \pm\infty$. Then $R = \bigcup_{t\in\mathbb{R}} \Phi_t(C) = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, a contradiction. □

## 4.3 Proof of Theorem C

In this section we assume, besides global attraction, that $\mathbf{0}$ is not positively stable and $\Phi$ has finitely many separators.

Let $\mathcal{X}$ be a separator configuration for $\Phi$, fix a small circle $C$ around $\mathbf{0}$ and let $X = \Delta_\Phi(\mathcal{X}, C)$ be the configuration of $\Phi$ in $C$. Also, fix an orientation $\Theta$ (counterclockwise or clockwise) in $\mathbb{R}^2$ and a heteroclinic separator $\Sigma$ in $\mathcal{X}$ (such an orbit exists because of Proposition 4.14(ii)). Let $q$ be the $\omega$-point of $\Sigma$ in $C$. Find disjoint open arcs $J, J' \subset C$ having $q$ as their common endpoint (small enough so that they do not contain any points from $X$), take points $p \in J$, $p' \in J'$, and assume that they are labelled so that the orientation of $(p, q, p')$ in $C$ is that given by $\Theta$ (that is, $(p, q, p')$ is positive if and only if $\Theta$ is the counterclockwise orientation). Finally, after removing $J'$ from $C$, we get an arc $A$ with endpoints $a$ (the other endpoint of $J'$) and $q$, and order the points from $A$ in the natural way so that $a < q$.

We call positive (negative) homoclinic orbits *even* when $\Theta$ is the counterclockwise (clockwise) orientation, and *odd* when $\Theta$ is the clockwise (counterclockwise) orientation. Thus, a homoclinic orbit from $\mathcal{X}$ is even if and only if its $\alpha$-point $v$ and its $\omega$-point $w$ satisfy $v < w$. By convention, all heteroclinic orbits are even. We say that two orbits have the *same parity* when both are even or both are odd.

According to Remark 4.3 and, again, Proposition 4.14(ii), all standard regions are indeed strips, so we will call them *standard strips*. Recall (Remark 4.3) that any standard strip must be either heteroclinic or homoclinic. By Lemma 4.8, the boundary of any heteroclinic standard strip $\Omega$ consists of (apart from $\mathbf{0}$) two heteroclinic separators (or just $\Sigma$, when $\Omega = R \setminus \Sigma$ is the union set of all heteroclinic orbits except $\Sigma$) and several (possibly zero) maximal homoclinic separators, when $\Omega$ is called *elementary* if and only if this last set is empty. Likewise, the boundary of a homoclinic standard strip $\Omega$ consists of, apart from $\mathbf{0}$, a homoclinic separator $\Gamma$ enclosing it and possibly some others, all of them less than $\Gamma$ in the $\prec$-ordering, when we again call $\Omega$ *elementary* if this last family is empty. Note that is quite possible for a standard strip to be elementary, but at least one heteroclinic standard strip cannot be elementary (otherwise $\Phi$ would have no homoclinic separators, and consequently all its regular orbits would be heteroclinic, contradicting Proposition 4.14(i)).

*Remark* 4.15. The following statements are easy to prove:

- a heteroclinic standard strip is elementary if and only if it is solid;
- a homoclinic standard strip is elementary if and only if the restriction of $\Phi$ to its closure is topologically equivalent to the flow induced by the "elliptic vector field" $f_4(x, y) = (x^2 - 2xy, xy - y^2)$ on the union set $A'_4$ of all orbits intersecting the diagonal arc $\{(x, x), 0 \le x \le 1\}$.

*Remark* 4.16. If a regular homoclinic separator $\Gamma$ is minimal, that is, $E(\Gamma)$ an elementary homoclinic standard strip, then there is an elliptic sector intersecting $E(\Gamma)$ (Remark 4.15). Thus, due to Remark 4.2, if $\mathbf{0}$ is not positively stable, and the finite sectorial decomposition property holds, then the decomposition must include both an elliptic and a hyperbolic sector.

There are two natural ways to associate an orbit from $\mathcal{X}$ to each standard strip $\Omega$ of $\Phi$. Firstly, $\gamma'(\Omega)$ will denote the orbit from $\mathcal{X}$ included in $\Omega$. Next, $\gamma(\Omega)$ will denote (when $\Omega$ is homoclinic) the separator $\Gamma \subset \operatorname{Bd}\Omega$ enclosing $\Omega$, and (when $\Omega$ is heteroclinic) the heteroclinic separator $\Gamma \subset \operatorname{Bd}\Omega$ whose $\omega$-point $w$ (in $C$, and then in $A$) satisfies $v < w$, $v$ being the $\omega$-point of $\gamma'(\Omega)$ Note that $\mathcal{X}$ consists of all orbits $\gamma(\Omega), \gamma'(\Omega)$ together with $\mathbf{0}$. Also, observe that $\gamma'(\Omega)$ decomposes $\Omega$ into two components $\Omega_l$ and $\Omega_u$, $\Omega_u$ being the component of $\Omega \setminus \gamma'(\Omega)$ including $\gamma(\Omega)$ in its boundary (an ambiguity arises in the case $\Omega = R \setminus \Sigma$, where $\Omega_u$ consists of the orbits whose $\omega$-points are greater than the $\omega$-point of $\gamma'(\Omega)$).

**Lemma 4.17.** *Let $\Omega$ be a standard strip and let $\Gamma$ be a regular orbit in $\operatorname{Bd}\Omega$. Then $\Gamma$ has the same parity as $\gamma'(\Omega)$ if and only if either $\Gamma = \gamma(\Omega)$ or $\Gamma \in \operatorname{Bd}\Omega_l$.*

*Proof.* We present the proof under the hypothesis that $\Omega$ is a heteroclinic strip whose boundary includes two heteroclinic orbits, $\gamma(\Omega)$ and $\gamma''(\Omega)$. The case when $\Omega$ is heteroclinic but $\Sigma$ is the only heteroclinic separator of $\Phi$, and the homoclinic case, can be dealt with in analogous fashion. We will also assume that the fixed orientation $\Theta$ is counterclockwise so the even (respectively odd) homoclinic orbits coincide with the positive (respectively negative) ones.

If $\Omega$ is elementary, then there is nothing to prove: both $\Gamma$ and $\gamma'(\Omega)$ are heteroclinic and consequently even. Otherwise, let $\Gamma_1, \ldots, \Gamma_j$ be the maximal homoclinic orbits in $\operatorname{Bd}\Omega$, where these orbits are labelled in such a way that if $q_1, \ldots, q_j$ are the corresponding $\omega$-points, then $q_1 < \cdots < q_j$ (in $A$). The corresponding $\alpha$-points will be denoted by $p_k$, $1 \le k \le j$. Finally, let $u$, $v$ and $w$ be the $\omega$-points of $\gamma''(\Omega)$, $\gamma'(\Omega)$ and $\gamma(\Omega)$, respectively (so $u < v < w$). We can assume, without loss of generality that there are small subarcs of $C$, neighbouring all these points, which are transversal to the flow.

Let $1 \le k \le j - 1$. We claim that it is not possible that $\Gamma_k$ is negative and $\Gamma_{k+1}$ is positive. Assume by contradiction $q_k < p_k < p_{k+1} < q_{k+1}$. Find points $p_k < b < b' < p_{k+1}$

in $C$, very close to $p_k$ and $p_{k+1}$, respectively, so that $T = \{t \in C : p_k \le t \le b\}$, $T' = \{t \in C : b' \le t \le p_{k+1}\}$ are transversal to the flow. Also, let $Q = \{t \in C : b \le t \le b'\}$. Since $\Gamma_k$ is negative, backward semiorbits starting from points from $T \setminus \{p_k\}$ enter the disk $D$ enclosed by $C$ and, since $\Gamma_{k+1}$ is positive, then they escape from the disk through $Q$. Accordingly, take a a decreasing sequence $(b_n)_{n=1}^{\infty}$ in $T \cap \Omega$ tending to $p_k$ and find maximal semiorbits $\varphi(a_n, b_n)$ fully included in $D$, when observe that the sequence $(a_n)_n$, besides lying in $Q$, is increasing. Call $a^*$ its limit. Clearly, $a^* \in \operatorname{Cl}\Omega$. Since the full forward orbit $\varphi(a^*, +)$ lies in $D$, and $\Gamma_k$ and $\Gamma_{k+1}$ are consecutive, we easily get that, in fact, $a^* \in \Omega$ and there is a solid heteroclinic strip $S$ neighbouring $a^*$. This is impossible because points $b_n$ belong to $S$ if $n$ is large enough, hence $\Gamma_k \subset \operatorname{Bd} S$.

Further, if $\Gamma_k$ and $\Gamma_{k+1}$ have the same sign, then $\gamma'(\Omega)$ cannot lie between them. In fact, assume, say, $q_k < p_k < v < q_{k+1} < p_{k+1}$, take $b$, $T$ and $(b_n)_n$ as before. but consider now $Q = \{t \in C : b \le t \le v\}$. Find similarly the points $a_n$ and $a^*$ in $Q$ to obtain the analogous contradiction. We prove that if $\Gamma_1$ is positive, then $\gamma'(\Omega)$ cannot lie between $\gamma''(\Omega)$ and $\Gamma_1$, and if $\Gamma_j$ is negative, then $\gamma'(\Omega)$ cannot lie between $\Gamma_j$ and $\gamma(\Omega)$, in the same way.

As a conclusion, we get that either (a) all orbits $\Gamma_k$ are positive and $\gamma'(\Omega)$ lies between $\Gamma_j$ and $\gamma(\Omega)$, or (b) all orbits $\Gamma_k$ are negative and $\gamma'(\Omega)$ lies between $\gamma''(\Omega)$ and $\Gamma_1$, or (c) there is $1 \le l \le j-1$ such that all orbits $\Gamma_k$ with $k \le l$ are positive, all orbits with $k > l$ are negative, and $\gamma'(\Omega)$ lies between $\Gamma_k$ and $\Gamma_{k+1}$. This implies the lemma. $\square$

We say that a finite, nonempty set $V$ of vectors of positive integers is *complete* when, for any $(i_1, \ldots, i_l) \in V$, we have $(i_1, \ldots, i_m) \in V$ for every $1 \le m \le l$, and $(i_1, \ldots, i_{l-1}, i) \in V$ for every $1 \le i \le i_l$. If $v \in V$, then we denote by $\lambda(v)$ the largest number $j$ such that $(v, j) \in V$, $\lambda(v) = 0$ meaning that there is no $j$ such that $(v, j) \in V$. Likewise, $\lambda(\emptyset)$ stands for the largest number $t$ such that $(t) \in V$. Of course we should write $\lambda_V$ instead of $\lambda$ (and similarly $\rho_L, \sigma_L$ instead of $\rho, \sigma$ below) to emphasize that this map depends on $V$, but hopefully this will not lead to confusion.

Let $\mathbb{M} = \{n/3 : n = 0, 1, 2, \ldots\}$.

**Definition 4.18.** We say that a set $L$ of vectors of numbers from $\mathbb{M}$ is *feasible* with *base* a complete set $V$ if its elements have the structure $(v, k)$, with $v \in V$ and $k \in \mathbb{M}$, and the following conditions hold:

(i) for each $(i) \in V$ there are exactly two elements in $L$: $(i, \lambda(i) + 1)$ and $(i, s + 2/3)$ for some integer $s = \sigma(i)$, $0 \le s \le \lambda(i)$;

(ii) for each $v \in V$ with length at least 2 there are exactly four elements in $L$: $(v, 0)$, $(v, \lambda(v) + 1)$, and $(v, r + 1/3)$, $(v, s + 2/3)$ for some integers $r = \rho(v)$, $s = \sigma(v)$, $0 \le r \le s \le \lambda(v)$;

(iii) $(i, \lambda(i)+2/3)$ and $(i+1, 2/3)$ cannot simultaneously belong to $L$ (where we mean $i+1=1$ when $i=\lambda(\emptyset)$);

(iv) if $\lambda(v)=1$, then $(v, 1/3)$, $(v, 5/3)$, $(v, 1, 1/3)$ and $(v, 1, \lambda(v,1)+2/3)$ cannot simultaneously belong to $L$.

Note that property (iii) above implies that $\lambda(i) \geq 1$ for some $i$, hence $V$ contains at least one sequence of length 2. If $V$ is the base of a feasible set $L$, then we assign a parity (even or odd) to each $v \in V$ as follows. All vectors of length 1 in $V$ have parity even. If $(i) \in V$, then we assign even or odd parity to $(i, j)$ depending on whether $j \leq \sigma(i)$ or not. Inductively, once the parity of $v \in V$ is established, we assign to $(v, j)$ the same parity as $v$, or the other one, depending on whether $\rho(v) < j \leq \sigma(v)$ or not. Finally, if $w = (v, h) \in L$, then we say that $w$ is an *$\alpha$-vector* if either $v$ is even and $h = 0$ or $h = \rho(v) + 1/3$, or $v$ is odd and $h = \lambda(v) + 1$ or $h = \sigma(v) + 2/3$. Otherwise, we say that $w$ is a *$\omega$-vector*.

We next explain how to associate canonically a feasible set $L$ to $\Phi$. To construct the base $V$ we proceed inductively, biunivocally associating to each standard strip $\Omega$ (and the $\omega$-point of $\gamma(\Omega)$) a vector from $V$. First of all, order the heteroclinic standard strips of $\Phi$ as $\Omega_1, \ldots, \Omega_t$, this meaning that the corresponding $\omega$-points $q_i$ of the orbits $\gamma(\Omega_i)$, $1 \leq i \leq t$, satisfy $q_1 < \ldots < q_t$. Then the 1-length vectors from $V$ will be those of the type $(i)$, $1 \leq i \leq t$. If, additionally, the strip $\Omega_i$ is not elementary, and $\Omega_{i,1}, \ldots, \Omega_{i,j}$ are the homoclinic standard strips $\Omega$ such that $\gamma(\Omega) \subset \operatorname{Bd} \Omega_i$ (again assuming $q_{i,1} < \ldots < q_{i,j}$ for their corresponding $\omega$-points), then we add the 2-vectors $(i, k)$ to $V$, $1 \leq k \leq j$. In general, if a vector $v$ has been added to $V$, with corresponding standard strip $\Omega_v$, and $\Omega_v$ is not elementary, then we order as before the homoclinic standard strips $\Omega$ such that $\gamma(\Omega) \subset \operatorname{Bd} \Omega_v$, call them $\Omega_{v,1}, \ldots, \Omega_{v,j'}$ (so that $q_{v,1} < \ldots < q_{v,j'}$ for the corresponding $\omega$-points), and add the vectors $(v, k)$, $1 \leq k \leq j'$, to $V$. Clearly, the set $V$ so defined is complete.

Now we define $L$ (and biunivocally associate to its vectors all points from $X$). We just must explain how to choose the numbers $\sigma(i)$ and the pairs $\rho(v), \sigma(v)$ in Definition 4.18(i) and (ii), and then check that (iii) and (iv) hold. As for the first numbers, let (with the notation of the previous paragraph) $1 \leq i \leq t$. Then $s = \sigma(i)$ is the largest number such that $q_{i,s} < y_i$, $y_i$ being the $\omega$-point of $\gamma'(\Omega_i)$ (or $s = 0$ if $\Omega_i$ is elementary or no such number exists, that is, $y_i < q_{i,j}$ for all $j$). Also, we redefine the points $y_i$ and $q_i$ as $c_{i,\sigma(i)+2/3}$ and $c_{i,\lambda(i)+1}$, respectively. In the general case we denote by $x_v$ and $y_v$ the $\alpha$- and $\omega$-points of $\gamma'(\Omega_v)$ when this orbit is even, reversing the notation when $\gamma'(\Omega_v)$ is odd, and take $r = \rho(v)$ and $s = \sigma(v)$ as the largest numbers satisfying $q_{v,r} < x_v$ and $q_{v,s} < y_v$, respectively (or $r = s = 0$ when $\Omega_v$ is elementary, and $r = 0$ or $s = 0$ when the corresponding number does not exist). Finally, we redenote $x_v$ and $y_v$ as $c_{v,\rho(v)+1/3}$ and $c_{v,\sigma(v)+2/3}$, while $c_{v,0}$ and $c_{v,\lambda(v)+1}$ stand for the $\alpha$- and $\omega$-points (or conversely in the odd case) of $\gamma(\Omega_v)$.

| $V$ | $L$ |
|---|---|
| $(1)$ | $(1,2)$, $(1,\frac{5}{3})$ |
| $(1,1)$ | $(1,1,0)$, $(1,1,2)$, $(1,1,\frac{1}{3})$, $(1,1,\frac{2}{3})$ |
| $(1,1,1)$ | $(1,1,1,0)$, $(1,1,1,1)$, $(1,1,1,\frac{1}{3})$, $(1,1,1,\frac{2}{3})$ |

Table 4.1: The elements of the feasible set $L$ and its base $V$ from the left flow of Figure 4.1.

| $V$ | $L$ |
|---|---|
| $(1)$ | $(1,2)$, $(1,\frac{2}{3})$ |
| $(1,1)$ | $(1,1,0)$, $(1,1,1)$, $(1,1,\frac{1}{3})$, $(1,1,\frac{2}{3})$ |
| $(2)$ | $(2,1)$, $(2,\frac{2}{3})$ |
| $(3)$ | $(3,2)$, $(3,\frac{5}{3})$ |
| $(3,1)$ | $(3,1,0)$, $(3,1,1)$, $(3,1,\frac{1}{3})$, $(3,1,\frac{2}{3})$ |

Table 4.2: The elements of the feasible set $L$ and its base $V$ from the right flow of Figure 4.1 ($\Sigma$ is the "upper" heteroclinic separator.

We claim that (iii) in Definition 4.18 holds. Indeed if, say, both $(i, \lambda(i) + 2/3)$ and $(i+1, 2/3)$ belong to $L$ for some $i$, the orbits $\gamma'(\Omega_i)$ and $\gamma'(\Omega_{i+1})$ would bound, together with $\mathbf{0}$, a solid strip (Remark 4.12). Since this strip includes the separator $\gamma(\Omega_i)$, we get a contradiction.

Assume now that Definition 4.18(iv) does not hold, that is, there is $v \in V$ with $\lambda(v) = 1$ such that all vectors $(v, 1/3)$, $(v, 5/3)$, $(v, 1, 1/3)$ and $(v, 1, \lambda(v,1)+2/3)$ belong to $L$. Then, again by Remark 4.12, the orbits $\gamma'(\Omega_v)$, $\gamma'(\Omega_{v,1})$ bound, together with $\mathbf{0}$, a solid strip including $\gamma(\Omega_{v,1})$, which is impossible.

Thus we have shown that $L$ is feasible. Although $L$ has been constructed with the help of the circle $C$, it depends only on $\Theta$ and $\Sigma$. We call it the *canonical feasible set* associated with $\Phi$, the orientation $\Theta$ and the separator $\Sigma$.

As some examples, we present in Tables 4.1 and 4.2 the feasible sets associated with the flows on Figure 4.1 under the counterclockwise orientation.

*Remark* 4.19. The simplest feasible set

$$L = \{(1,5/3), (1,2), (1,1,0), (1,1,1/3), (1,1,2/3), (1,1,1)\}$$

(equivalent to

$$L = \{(1,2/3), (1,2), (1,1,0), (1,1,1/3), (1,1,2/3), (1,1,1)\}$$

after reversing the orientation) correspond to the case when there are exactly three separators (one heteroclinic, another one regular homoclinic, and the singular point), which occurs when "$\prec$" is a total ordering in $\mathcal{H}$ ($\mathbf{0}$ becoming an elliptic saddle for the flow).

Observe that the bijection from $L$ to $X$ given by $w \mapsto c_w$ preserves *orders* (when the lexicographical order is used in $L$), *orbits* (that is, two points $c_w$ and $c_{w'}$ belongs to the same orbit if and only if $w = (v, h)$ and $w' = (v, h')$ for some $v \in V$ and $h + h'$ is an integer) and *directions* (that is, $w$ is a $\omega$-vector if and only if $c_w$ is a $\omega$-point; this follows from Lemma 4.17, which implies that the parity of $v \in V$ is the same as that of $\gamma(\Omega_v)$ and $\gamma'(\Omega_v)$). There are many feasible sets $L'$ which can be bijectively mapped onto $X$ so that ordering is preserving: since both orderings are total, one just needs that both cardinalities of $L$ and $L'$ are the same. As it turns out, if orbits are preserved, then directions are preserved as well:

**Lemma 4.20.** *If $L'$ is feasible, and there is a bijection $\psi : L' \to X$ preserving orders and orbits, then $L' = L$.*

*Proof.* Let $V'$ the base of $L'$ and redenote $\lambda_{V'} = \lambda'$, $\rho_{L'} = \rho'$, $\sigma_{L'} = \sigma'$. Since $\psi$ preserves orbits, it maps vectors $(i', \lambda'(i')+1)$ and $(i', \sigma'(i')+2/3)$ to $\omega$-points of heteroclinic orbits, and pairs $(v', 0)$ and $(v', \lambda'(v')+1)$, as well as pairs $(v', \rho'(v')+1/3)$ and $(v', \sigma'(v')+2/3)$, to pairs of points of homoclinic orbits. Since orders are preserved as well, we get that vectors $(i', \lambda'(i') + 1)$ are precisely those mapped to heteroclinic separators, and deduce that vectors of lengths 1 and 2 of $V$ and $V'$, as well as vectors of length 2 of $L$ and $L'$, are the same. Now, as the reader will easily convince himself, to prove the lemma we just have to show this: pairs $(v', 0)$ and $(v', \lambda'(v')+1)$ are exactly those mapped to homoclinic separators.

Assume, to arrive at a contradiction, that $(v', 0)$ and $(v', \lambda'(i')+1)$ are mapped to one of the orbits $\gamma'(\Omega_v)$ of $\mathcal{X}$. Since $\mathcal{X}$ has no orbits between $\gamma'(\Omega_v)$ and the orbits $\gamma(\Omega_{v,k})$ (regarding the order "$\prec$"), it is clear that $(v', \rho'(v') + 1/3)$ and $(v', \sigma'(v') + 2/3)$ must be mapped to one of the orbits $\gamma(\Omega_{v,k})$ (in particular, $v$ cannot have maximal length in $V$). Similarly, if $(v', \rho'(v')+1/3)$ and $(v', \sigma'(v')+2/3)$ are mapped to an orbit $\gamma(\Omega_w)$, the pair which is mapped to $\gamma'(\Omega_w)$ must be of the type $(w', 0)$ and $(w', \lambda'(w') + 1)$, because the orbit corresponding to $(w', \rho'(w') + 1/3)$ and $(w', \sigma'(w') + 2/3)$ is $\prec$-less than that corresponding to $(w', 0)$ and $(w', \lambda'(w') + 1)$, and there are no orbits of $\mathcal{X}$ between $\gamma(\Omega_w)$ and $\gamma'(\Omega_w)$. We could thus proceed indefinitely, contradicting the finiteness of $\mathcal{X}$. $\square$

*Proof of Theorem C.* The statement (i)$\Rightarrow$(ii) is obvious (recall Proposition 4.14).

Let us show (ii)$\Rightarrow$(iii). Fix small circles $C, C'$ around $\mathbf{0}$ and let $\mu : C \to C'$ be a homeomorphism preserving orbits for $\xi$. Use the hypothesis to find heteroclinic separators $\Sigma$ and $\Sigma'$ such that $\xi(\Sigma) = \Sigma'$, fix $\Theta$ as the counterclockwise orientation, and take $\Theta'$ as the counterclockwise or the clockwise orientation depending on whether $\mu$ preserves or reverses the orientation. Construct the canonical feasible sets $L$ and $L'$ associated with

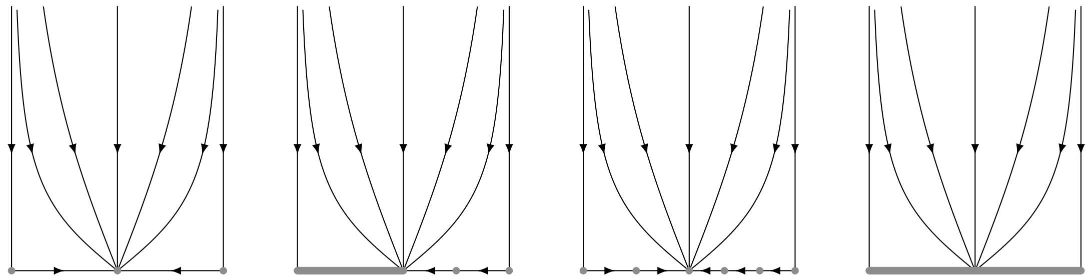

Figure 4.3: From left to right: phase portraits of $f$, $f_{0,2}$,$f_{2,5}$ and $f_{0,0}$.

them, and the corresponding bijections $\psi : L \to X$, $\psi' : L' \to X'$ to the configurations of $\mathcal{X}$ and $\mathcal{X}'$ preserving orders, orbits and directions. Although the hypothesis does not state that $\mu$ preserves directions for $\xi$, we get that $\mu^{-1} \circ \psi' : L' \to X$ preserves orders and orbits anyway. Now Lemma 4.20 applies and (iii) follows.

Finally, to prove (iii)⇒(i), let again $C, C'$ be small circles around $\mathbf{0}$, denote the configurations of $\mathcal{X}$ and $\mathcal{X}'$ in these circles by $X$ and $X'$, and find arcs $A \subset C$ and $A' \subset C'$ containing all points of $X$ and $X'$ and having $q$ and $q'$, the $\omega$-points of $\Sigma$ and $\Sigma'$, as their upper endpoints (after using the respective orientations $\Theta$ and $\Theta'$). According to the hypothesis, there are a feasible set $L$ and bijections $\psi : L \to X$, $\psi' : L \to X'$ preserving orders, orbits and directions, and hence a bijection $\xi : \mathcal{X} \to \mathcal{X}'$ and a homeomorphism $\mu : C \to C'$ preserving orbits and directions for $\xi$. Then, as explained in Subsection 4.1.2, there is a plane homeomorphism preserving the separator configuration orbits, which turns out to preserve the time directions as well. Hence $\Phi$ and $\Phi'$ are topologically equivalent by Theorem 4.4. □

## 4.4 Proof of Theorem D

Let $0 \leq s \leq j$ be non-negative integers. We define a $C^1$-vector field $f_{s,j}$ as follows. We start from $f(x, y) = (x(x^2 - 1), -y)$. As easily checked, the phase portrait of (the associated local flow with) $f$ in the semi-band $[-1, 1] \times [0, \infty)$ (the only sector we are interested in) consists of three singular points ($\mathbf{0}$, $(-1, 0)$ and $(1, 0)$), two horizontal orbits in the $x$-axis going to $\mathbf{0}$ as time goes to $\infty$, and three vertical orbits on the semi-lines $x = -1, 0, -1$, each converging in positive time to the corresponding singular point. All other orbits go to $\mathbf{0}$ as $t \to \infty$. Next, let $\kappa(x)$ be a non-negative $C^1$-function vanishing at points $x = -i/s$, $0 \leq i \leq s$ (or at the whole interval $[-1, 0]$ if $s = 0$), at points $x = i/(j - s)$, $0 \leq i \leq j - s$ (or at the whole interval $[0, 1]$ if $s = j$), and at no other points. Then we define $f_{s,j}(x, y) = (\kappa(x) + y^2)f(x, y)$, thus adding new singular points in the $x$-axis and leaving unchanged the upper orbits. Figure 4.3 exhibits the phase portrait of $f_{s,j}$ for different values of $s$ and $j$.

Now, let $0 \leq r \leq s \leq j$ be non-negative integers and define $C^1$-vector fields $g^+_{r,s,j}, g^-_{r,s,j}$

| Directions | Regions |
|---|---|
| $x' < 0,\ y' > 0$ | $V_1 = \{(x, y) \in V : x < -1 + (1-y)^2/2\}$ |
| $x' > 0,\ y' > 0$ | $V_2 = \{(x, y) \in V : -1 + (1-y)^2/2 < x < 0\}$ |
| $x' > 0,\ y' < 0$ | $V_3 = \{(x, y) \in V : 0 < x < 1 - (1-y)^2/2\}$ |
| $x' < 0,\ y' < 0$ | $V_4 = \{(x, y) \in V : 1 - (1-y)^2/2 < x\}$ |

Table 4.3: Directions of the vector field for the system (4.1).

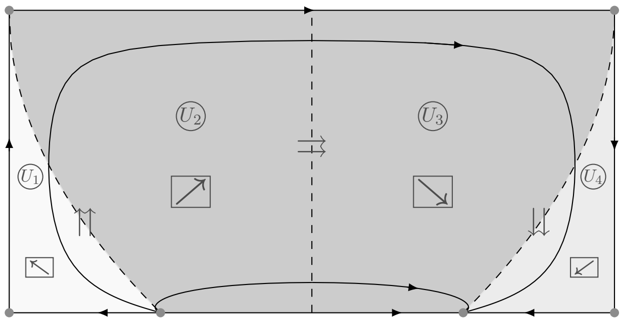


Figure 4.4: Phase portrait of $g$.

as follows. This time our starting point is

$$g(x, y) = \left( (x^2 - 1) \left( x^2 - \left( 1 - \frac{(1-y)^2}{2} \right)^2 \right), y(y-1)x \right)$$

and we are interested in its phase portrait in the rectangle $[-1, 1] \times [0, 1]$. We have six singular points: $(-1, 0)$, $(1, 0)$, $(-1/2, 0)$, $(1/2, 0)$, $(-1, 1)$ and $(1, 1)$. The boundary of the rectangle is invariant for the flow, hence consisting of the singular points and six regular orbits, all clockwise oriented by the flow except that connecting $(-1/2, 0)$ and $(1/2, 0)$. Additional isoclines exist at the $y$-axis (for the horizontal direction of the flow) and the parabolas $x = \pm(1-(1-y)^2/2)$ (for the vertical direction of the flow). This three isoclines divide the interior of the rectangle $V = (-1, 1) \times (0, 1)$ in four regions $V_i$, $1 \leq i \leq 4$, where the flow has a well-defined direction: see Table 4.3 and Figure 4.4.

This already ensures that all orbits through points in $V$ crossing the $y$-axis go to $(-1/2, 0)$ (respectively, $(1/2, 0)$) as time goes to $-\infty$ (respectively, $\infty$).

As it happens, this completes the phase portrait because in fact all interior orbits cross the $y$-axis. To prove this we must discard the existence of full orbits in the region to the right of the isocline $x = 1-(1-y)^2/2$ or, equivalently (because of the symmetry properties of the vector field) in the region to the left of the isocline $x = -1 + (1-y)^2/2$. Now, in order to prove that there are no full orbits to the right of $x = 1 - (1-y)^2/2$, it clearly suffices to show that the vector field crosses from left to right all lines $y = 1 + a(x - 1)$,

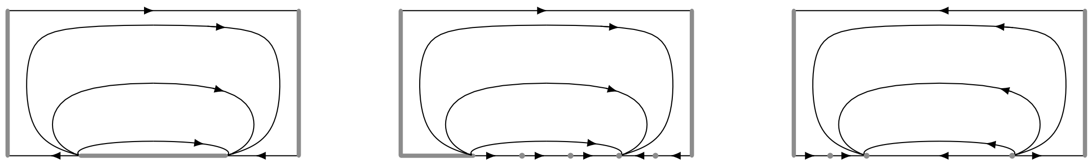

Figure 4.5: Phase portraits of $g^+_{1,1,2}$ (left), $g^+_{0,3,5}$ (center) and $g^-_{2,3,4}$ (right).

$a > 0$, in the square $(1/2, 1) \times (1/2, 1)$, that is, $ag_1(1-t, 1-at) - g_2(1-t, 1-at) > 0$ whenever $0 < t < 1/2$ and $0 < at < 1/2$, when we mean $g = (g_1, g_2)$. Since

$$\begin{aligned}\frac{ag_1(1-t,1-at) - g_2(1-t,1-at)}{at} &= 1 + 3t - at - 4t^2 + at^2 - 2a^2t^2 \\ &\quad +(1+a^2)t^3 + \frac{a^4t^4}{2} - \frac{a^4t^5}{4} \\ &> 1 + 3t - \frac{1}{2} - 2t + at^2 - \frac{1}{2} \\ &\quad +(1+a^2)t^3 + \frac{a^4t^4}{2} - \frac{t}{64} \\ &= \frac{63t}{64} + at^2 + (1+a^2)t^3 + \frac{a^4t^4}{2} \ > \ 0,\end{aligned}$$

we are done.

Let $\kappa(x)$ be a non-negative $C^1$-function vanishing at points $x = -1 + i/(2r)$, $0 \le i \le r$ (or at the whole interval $[-1, -1/2]$ if $r = 0$), at points $x = -1/2 + i/(s-r)$, $0 \le i \le s-r$ (or at the whole interval $[-1/2, 1/2]$ if $r = s$), at points $x = 1/2 + i/(2j-2s)$, $0 \le i \le j-s$ (or at the whole interval $[1/2, 1]$ if $s = j$), and at no other points. Then we define $g^+_{r,s,j}(x,y) = (\kappa(x) + y^2)(1-x^2)g(x,y)$. In this way, we add some new singular points at the $x$-axis, and all points from both vertical borders of the rectangle become singular as well, yet the inner orbits remain the same. Finally we put $g^-_{r,s,j}(x,y) = -g_{r,s,j}(x,y)$, getting the same phase portrait with reversed time directions. Some examples of the phase portraits of these vector fields are shown in Figure 4.5.

Let $L$ be a feasible set with base $V$. We are ready to explain how to construct a polynomial flow $\Phi$ whose associated feasible set, after fixing the counterclockwise orientation and choosing an appropriate heteroclinic separator of $\Phi$, is exactly $L$.

Let $n$ be the length of the largest sequence in $V$ and recall that $n \ge 2$. Also, let $t = \lambda(\emptyset) \ge 1$. Firstly, we define a vector field $F$ on $\mathbb{R}^2$ by gluing (after appropriate translations and dilatations) some vectors fields $f_{r,j}, g^+_{r,s,j}, g^-_{r,s,j}$ (and the null vector field) as prescribed by $L$.

To begin with, if $(i) \in V$, then we glue at the semi-band $[i-1, i] \times [0, \infty)$ the vector field $f_{\sigma(i),\lambda(i)}$ (that is, $f_{\sigma(i),\lambda(i)}(2x - 2i + 1, y)$). Note that the way we defined the maps $f_{s,j}$ ensures that adjacent pieces glue well at the orbits $\Upsilon_i := \{i\} \times [0, \infty)$.

Now, the maximal compact intervals $I$ in $I_i := [i-i, i]$ such that $\operatorname{Int} I \times \{0\}$ contains no singular points will be denoted, from left to right, by $I_{i,1}, \dots, I_{i,\lambda(i)}$, the flow travelling to the right on $\Upsilon_{i,k} := I_{i,k} \times \{0\}$ if and only if $k \leq \sigma(i)$. Certainly, maximal compact intervals $N$ with $N \times \{0\}$ just consisting of singular points may exist; we call each of them a 0-level null interval.

After $F$ has been defined on $[0,t] \times [0,\infty)$, we define it in $[0,t] \times [-1,0)$. In the rectangles $N \times [-1,0)$, where $N$ is a 0-level null interval, we just define $F$ as zero; and at the rectangles $I_{i,k} \times [-1,0)$ we glue either the vector field $g^+_{\rho(i,k),\sigma(i,k),\lambda(i,k)}$ (more properly,

$$g^+_{\rho(i,k),\sigma(i,k),\lambda(i,k)}((2x-a-b)/(b-a), y+1)$$

with $I_{i,k} = [a,b]$) or the vector field $g^-_{\rho(i,k),\sigma(i,k),\lambda(i,k)}$ according to whether the flow in $\Upsilon_{i,k}$ goes to the right or to the left. Similarly as before, the maximal compact intervals $I$ in $I_{i,k}$ such that $\operatorname{Int} I \times \{-1\}$ contains no singular points will be denoted, ordered from left to right, $I_{i,k,1}, \dots, I_{i,k,\lambda(i,k)}$ (write also $\Upsilon_{i,k,k'} = I_{i,k,k'} \times \{-1\}$), and the flows travels on $\Upsilon_{i,k,k'}$ in the same direction as in $\Upsilon_{i,k}$ if and only if $\rho(i,k) < k' \leq \sigma(i,k)$. Any maximal compact interval $N$ such that $N \times \{-1\}$ consists of singular points will be called a 1-level null interval.

Proceeding in this way, we associate inductively to each vector $v \in V$ of length $m \geq 2$ an interval $I_v \in [0,t]$ (and the corresponding orbit $\Upsilon_v = I_v \times \{-m+2\}$), and define the $m$-level null intervals. Then we define $F$ as zero in $N \times [-m+1, -m+2)$ if $N$ is $m$-level null, or as $g^+_{\rho(v),\sigma(v),\lambda(v)}$ or $g^-_{\rho(v),\sigma(v),\lambda(v)}$ in $I_v \times [-m+1, -m+2)$ according to the direction of the flow on $\Upsilon_v$. Note that the full lowest segment $[0,t] \times \{-n+1\}$ is null, that is, all its points are singular.

Thus we have completed the definition of $F$ on $[0,t] \times [-n+1, \infty)$. Note that the map so defined is not locally Lipschitz (or even continuous) at the orbits $\Upsilon_v$; this can be easily arranged by multiplying $F$ by appropriate positive $C^1$-functions $\tau_v(x)$ in the corresponding semi-open rectangles $\operatorname{Int} I_v \times [-m+1, -m+2)$. We keep calling $F$ this modified map; note that, even so, it needs not be continuous at the singular points. To conclude the definition of $F$, we extend it periodically to the whole semi-plane $\mathbb{R} \times [-n+1, \infty)$ (that is $F(x,y) = F(x+kt, y)$ for any integer $k$) and define it as zero otherwise.

Before proceeding further, some additional notation must be given. First, let $\Upsilon'_i = \{i - 1/2\} \times [0,\infty)$, $i = 1, \dots, t$. Also, for any $v \in V$ with length $m \geq 2$, let $\Upsilon'_v$ be the orbit in $I_v \times (-m+1, -m+2)$ corresponding, after translation and dilatation, to the orbit of the vector field $g(x,y)$, passing through the point $(0, 1/2)$. Now it is easy to construct a poligonal arc $A$ with endpoints $(0,1)$ and $(t,1)$, consisting of alternate horizontal and vertical segments, so that:

- horizontal segments are of type $J \times \{-m + \epsilon_J\}$ for some compact interval $J$, some

| $V$ | $L$ |
|---|---|
| $(1)$ | $(1, \frac{5}{3})$, $(1, 4)$ |
| $(1, 1)$ | $(1, 1, 0)$, $(1, 1, \frac{1}{3})$, $(1, 1, \frac{2}{3})$, $(1, 1, 1)$ |
| $(1, 2)$ | $(1, 2, 0)$, $(1, 2, \frac{1}{3})$, $(1, 2, \frac{2}{3})$, $(1, 2, 1)$ |
| $(1, 3)$ | $(1, 3, 0)$, $(1, 3, \frac{1}{3})$, $(1, 3, \frac{2}{3})$, $(1, 3, 1)$ |
| $(2)$ | $(2, \frac{2}{3})$, $(2, 2)$ |
| $(2, 1)$ | $(2, 1, 0)$, $(2, 1, \frac{1}{3})$, $(2, 1, \frac{8}{3})$, $(2, 1, 3)$ |
| $(2, 1, 1)$ | $(2, 1, 1, 0)$, $(2, 1, 1, \frac{1}{3})$, $(2, 1, 1, \frac{2}{3})$, $(2, 1, 1, 1)$ |
| $(2, 1, 2)$ | $(2, 1, 2, 0)$, $(2, 1, 2, \frac{1}{3})$, $(2, 1, 2, \frac{2}{3})$, $(2, 1, 2, 1)$ |
| $(3)$ | $(3, \frac{2}{3})$, $(3, 1)$ |
| $(4)$ | $(4, \frac{5}{3})$, $(4, 2)$ |
| $(4, 1)$ | $(4, 1, 0)$, $(4, 1, \frac{4}{3})$, $(4, 1, \frac{5}{3})$, $(4, 1, 3)$ |
| $(4, 1, 1)$ | $(4, 1, 1, 0)$, $(4, 1, 1, \frac{1}{3})$, $(4, 1, 1, \frac{2}{3})$, $(4, 1, 1, 1)$ |
| $(4, 1, 2)$ | $(4, 1, 2, 0)$, $(4, 1, 2, \frac{1}{3})$, $(4, 1, 2, \frac{2}{3})$, $(4, 1, 2, 1)$ |

Table 4.4: The elements of the feasible set $L$ and its base $V$ from Figure 4.6.

$0 < \epsilon_J \leq 1$ and $0 \leq m < n$;

- any two such intervals $J, J'$ have at most one common point, and the union of all intervals $J$ is $[0, t]$;
- $A$ intersects each orbit $\Upsilon_i, \Upsilon'_i$ at exactly one point, and all other orbits $\Upsilon_v, \Upsilon'_v$ at exactly two points.

Observe that the bijection mapping $L$ to the set of these intersection points that preserves orders (hence mapping $(t, \lambda(t) + 1)$ to $(t, 1)$), also preserves orbits as previously meant, that is, every vector $(i, h)$ is mapped either to $A \cap \Upsilon_i$ or to $A \cap \Upsilon'_i$ and every pair of vectors $(v, h), (v, h')$ with $h + h'$ an integer is mapped either to $A \cup \Upsilon_v$ or to $A \cup \Upsilon'_v$.

Figure 4.6 illustrates the former construction starting from the feasible set $L$ described in Table 4.4. The dotted line indicates the arc $A$.

Let $\Xi : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be given by $\Xi(r, \theta) = e^{r + \mathbf{i}2\pi\theta/t}$. Although $F$ may not be continuous, the set $T$ of singular points of $F$ is closed and $F$ is locally Lipschitz in the region $O = \mathbb{R}^2 \setminus T$; hence, when restricted to $O$, it has an associated local flow which can be naturally carried to the region $U = \Xi(O)$ via $\Xi$: call $\Psi'$ this projected local flow on $U$. Let $\Psi$ be a flow on $\mathbb{R}^2_\infty$ with the same orbits and time orientations as $\Psi'$, and having singular points outside $U$, that is, at $K = \Xi(T) \cup \{\mathbf{0}\}$ and $\infty$. This flow induces in $\mathcal{Q} = \mathbb{R}^2_\infty / \sim_K$,

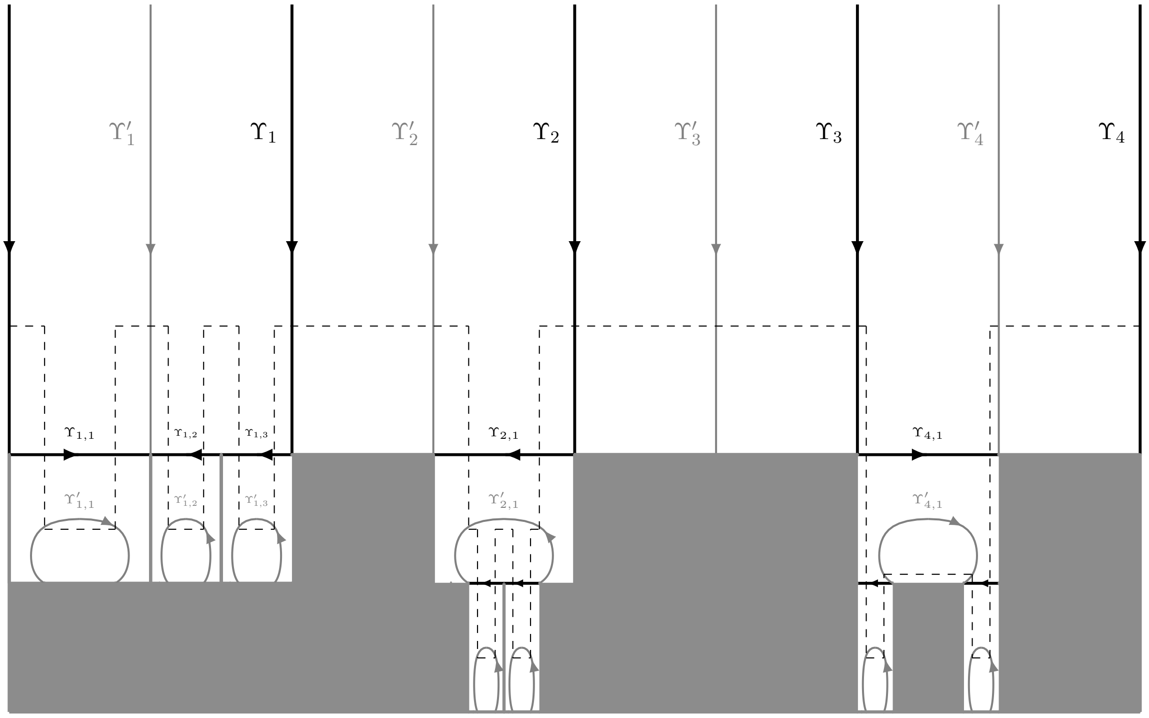


Figure 4.6: Constructing a polynomial flow from a feasible set.

in the natural way, a flow $\Psi_{\sim_K}$ with two singular points, $K$ (now an element of $\mathcal{Q}$) and $\infty$. Moreover, since $\mathbb{R}^2_\infty \setminus K$ is connected, there is a homeomorphism $H : \mathcal{Q} \to \mathbb{R}^2_\infty$ (Lemma 1.4), when we can assume $H(K) = \mathbf{0}$, $H(\infty) = \infty$. After carrying $\Psi_{\sim_K}$ to $\mathbb{R}^2_\infty$ via $H$, we get a flow $\Phi'$ on $\mathbb{R}^2_\infty$ having (when restricted to $\mathbb{R}^2$) $\mathbf{0}$ as its global attractor, its separator configuration consisting of $\mathbf{0}$ and the curves $(H \circ \Xi)(\Upsilon_v), (H \circ \Xi)(\Upsilon'_v)$, $v \in V$. Using $C = (H \circ \Xi)(A)$, now a circle around $\mathbf{0}$, choosing an appropriate orientation $\Theta$ in $\mathbb{R}^2$, and taking $\Sigma = (H \circ \Xi)(\Upsilon_t)$ (recall also Lemma 4.20), we get that $L$ is the canonical feasible set associated with $\Phi'$, $\Theta$ and $\Sigma$. Composing $H$ if necessary with a reversing order homeomorphism, we can in fact get $\Theta$ to the the counterclockwise orientation.

We are almost done. Indeed, since $\Phi'$ has finitely many unstable orbits, two singular points (the only possible $\alpha$-limit and $\omega$-limit sets of the flow) and no periodic orbits, [46, Lemma 4.1] (essentially, a corollary of Theorem 1.49 and the main results in [74]) implies that it is topologically equivalent to the associated flow with a polynomial vector field in $\mathbb{S}^2$ and then, as explained in Section 4.1, to a polynomial flow in $\mathbb{R}^2$. Figure 4.7 shows the resultant flow after collapsing the flow from Figure 4.6.

*Remark* 4.21. Since any flow having $\mathbf{0}$ as a global attractor and finitely many separators is topologically equivalent to a polynomial flow, and polynomial flows have the finite sectorial decomposition property, we get that finiteness of separators and sectors are, in fact, equivalent properties in this setting (compare to Remark 4.2).

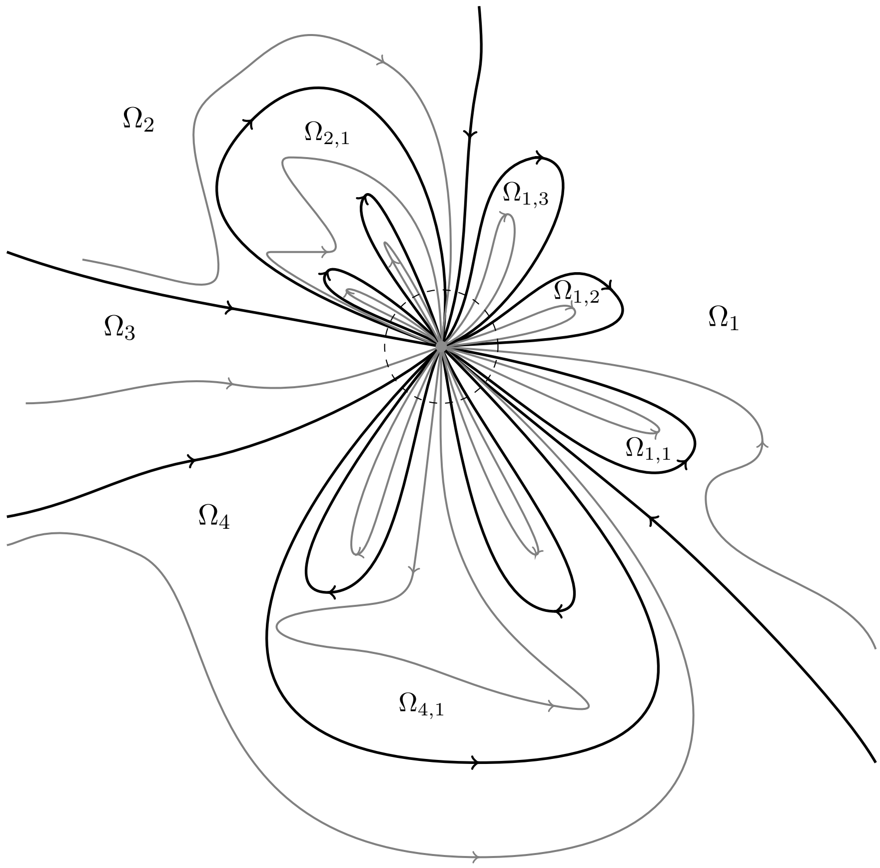


Figure 4.7: The phase portrait of the flow labelled by the feasible set from Table 4.4.

| Directions | Regions |
|---|---|
| $x' < 0$, $y' > 0$ | $U_1 = \{(x, y) : y > 0, y^2 + x^3 > 0\}$ |
| $x' < 0$, $y' < 0$ | $U_2 = \{(x, y) : y^2 + x^3 < 0, (1 + x^2)y + x^3 > 0\}$ |
| $x' > 0$, $y' < 0$ | $U_3 = \{(x, y) : (1 + x^2)y + x^3 < 0, y > 0\}$ |
| $x' > 0$, $y' < 0$ | $U_4 = \{(x, y) : y < 0, y^2 + x^3 < 0\}$ |
| $x' > 0$, $y' > 0$ | $U_5 = \{(x, y) : y^2 + x^3 > 0, (1 + x^2)y + x^3 < 0\}$ |
| $x' < 0$, $y' > 0$ | $U_6 = \{(x, y) : (1 + x^2)y + x^3 > 0, y < 0\}$ |

Table 4.5: Directions of the vector field for the system (4.1).

## 4.5 Proof of Theorem E

Since the polynomial $(1+x^2)^2+x^3$ has no real zeros, $\mathbf{0}$ is the only singular point of the associated local flow with (4.1). The isocline corresponding to the horizontal direction of the vector field is the union of the curves $y = 0$ and $y^2 + x^3 = 0$. Thus, the $x$-axis consists of $\mathbf{0}$ and two regular orbits (both going to $\mathbf{0}$ in positive time) and there are no periodic orbits, as they should enclose the singular point. The isocline corresponding to the vertical direction of the vector field is the curve $(1 + x^2)y + x^3 = 0$. Finally, the isoclines divide the plane in six regions $U_i$, $1 \leq i \leq 6$, where the flow has a well-defined direction: see Table 4.5 and Figure 4.8.

*Claim 1: The origin is a global attractor of* (4.1).

First of all, observe that orbits starting in $U_2$ go to $U_3$, and orbits starting in $U_3$ go to $\mathbf{0}$. Similarly, orbits starting in $U_4$ go to $U_5$, orbits starting in $U_5$ either go to $\mathbf{0}$ or to $U_6$, and orbits starting in $U_6$ go to $\mathbf{0}$. As a consequence, in order to prove the claim, it is enough to show that any orbit starting in $U_1$ meets the curve $y^2 + x^3 = 0$.

Let $P(x, y) = -((1 + x^2)y + x^3)^5$ and $Q(x, y) = y^2(y^2 + x^3)$ be the components of the vector field and put $U_1' = U_1 \cap \{(x, y) : y \geq 1\}$. Then we have

$$-1 \leq \frac{Q(x,y)}{P(x,y)} \leq 0 \qquad \text{for any } (x,y) \in U_1' \tag{4.2}$$

because if $x \geq 0$, then

$$Q(x, y) = y^4 + y^2x^3 \leq (1 + x^2)^5y^5 + 5(1 + x^2)^4y^4x^3 \leq |P(x, y)|,$$

while if $x \leq 0$, we use that $y \geq -x$ holds in $U_1'$ to get

$$Q(x, y) \leq y^4 \leq y^5 \leq (y + yx^2 + x^3)^5 = |P(x, y)|.$$

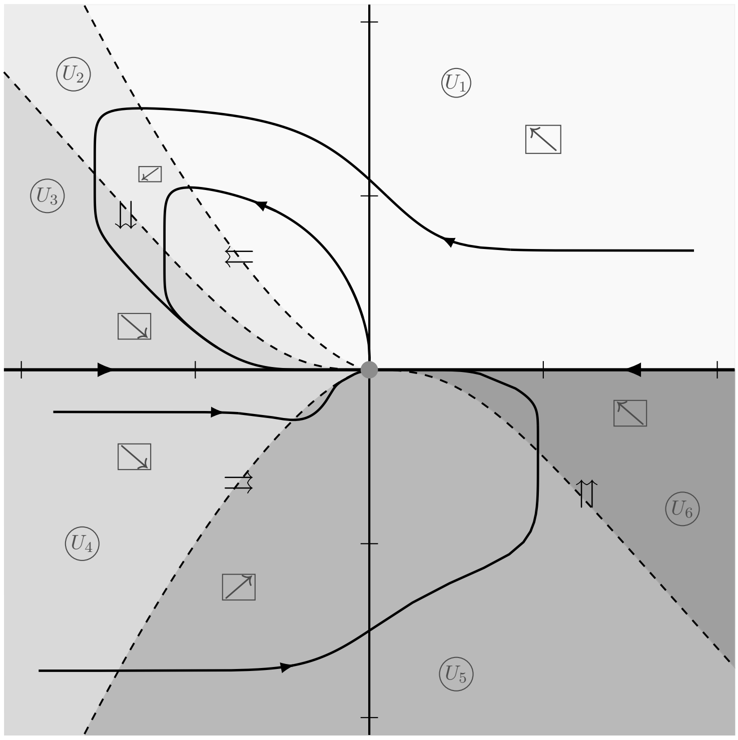


Figure 4.8: Phase portrait of $x' = -((1+x^2)y + x^3)^5$, $y' = y^2(y^2+x^3)$.

Now, realize that if an orbit starts in $U_1$, then either it crosses $y^2 + x^3 = 0$, or goes to $U_1'$. Therefore, to prove the claim, it suffices to show that if $(x_0, y_0) \in U_1'$, then the orbit (corresponding to the solution) $(x(t), y(t))$ of (4.1) starting at $x(0) = x_0$ and $y(0) = y_0$ meets $y^2 + x^3 = 0$. But, due to (4.2), we have $y'(t) \le -x'(t)$ and then $y(t) \le x_0 + y_0 - x(t)$ whenever the orbit stay in $U_1'$. In other words, the orbit lies below the line $y = x_0 + y_0 - x$ while staying in $U_1'$. Since this line intersects $y^2 + x^3 = 0$, Claim 1 follows.

*Claim 2: The origin is not positively stable for* (4.1).

Given any $y_0 > 0$, let $(x(t), y(t))$ be the orbit of (4.1) starting at $x(0) = 0$ and $y(0) = y_0$. According to Claim 1, this orbit must travel to $U_2$, then to $U_3$, and finally converge to $\mathbf{0}$. In particular, it meets the line $y = -2x$. Let $t_*$ be the (smallest) positive time for which $y(t_*) = -2x(t_*)$ and denote $Y(y_0) = y(t_*)$.

To prove the claim, it suffices to show that $Y(y_0) > 1/2$ (this bound is conservative; numerical estimations suggest that the optimal bound is approximately 0.831). We proceed by contradiction assuming $Y(y_0) \le 1/2$. Then $-1/4 \le x(t) \le 0$ for any $0 \le t \le t_*$.

For the sake of clarity, in this paragraph we assume $0 \le t \le t_*$ and shorten $x(t)$ as $x$ and $y(t)$ as $y$. Since $x \le 0$, we trivially have

$$y + \frac{x^3}{1+x^2} \le y + (-x)^{3/2}. \tag{4.3}$$

We assert that
$$y + \frac{x^3}{1+x^2} \leq 2\left(y - (-x)^{3/2}\right) \tag{4.4}$$
is true as well. Observe that (4.4) is equivalent to
$$2(1+x^2)(-x)^{3/2} + \frac{x^3}{\leq}(1+x^2)y$$
and, taking into account that $y \geq -2x$, a sufficient condition for this to happen is
$$(-2x(1+x^2) - x^3)^2 - (2(1+x^2)(-x)^{3/2})^2 \geq 0,$$
which is true indeed:
$$\begin{aligned}(-2x(1+x^2) - x^3)^2 - (2(1+x^2)(-x)^{3/2})^2 &= x^2(4+4x+12x^2+8x^3+9x^4+4x^5)\\ &\geq 4x^2(1+x+2x^3+x^5)\\ &\geq 4x^2\left(1-\frac{1}{4}-\frac{1}{32}-\frac{1}{1024}\right) \geq 0.\end{aligned}$$
Finally, we have
$$\frac{1}{(1+x^2)^5} \geq \frac{1}{(1+1/16)^5} = \frac{1048576}{1419857} > \frac{1}{2}. \tag{4.5}$$
Putting together (4.3), (4.4) and (4.5), we get
$$\frac{Q(x,y)}{P(x,y)} = -\frac{y^2(y+(-x)^{3/2})(y-(-x)^{3/2})}{(1+x^2)^5(y+x^3/(1+x^2))^5} \leq -\frac{1}{4y}.$$

As a consequence, for every $0 \leq t \leq t_*$, we have $2y'(t)y(t) \geq -x'(t)$ and therefore
$$y(t)^2 \geq y_0^2 - x(t) > -x(t),$$
that is, the orbit lies over the parabola $y^2 = -x$. Since this parabola intersects $y = -2x$ at the point $(-1/4, 1/2)$, we obtain the desired contradiction $Y(t_0) > 1/2$, and Claim 2 follows.

*Claim 3: The origin is an elliptic saddle for* (4.1)*.*

Let $R$ be the union set of all heteroclinic orbits of (4.1), that is, the closed lower half-plane (except $\mathbf{0}$) and all orbits intersecting the positive semi-$y$-axis. By Claims 1 and 2, $R$ is a radial region strictly included in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ (Proposition 4.14). Moreover, it is clear that this flow does not allow a pair of incomparable homoclinic orbits. Then $\operatorname{Bd} R = \Gamma \cup \{0\}$, $\Gamma$ being the only regular homoclinic separator of the flow (the other separators are the positive semi-$x$-axis and $\mathbf{0}$), and $\mathbf{0}$ is an elliptic saddle (Remark 4.19).

Claims 1, 2 and 3 complete the proof of Theorem E.

# Chapter 5

# On the $\omega$-limit sets for analytic flows on open sets of the sphere

In [43], V. Jiménez and J. Llibre characterized, up to topological deformation, the $\omega$-limit sets for analytic flows on the sphere, the plane and the projective plane. Unfortunately, there is a gap in the proof of the essential Lemma 4.6 there. The proof can be amended when the surface $S$ is either the sphere $\mathbb{S}^2$, the plane $\mathbb{R}^2$, the projective plane $\mathbb{P}^2$ or the projective plane minus one point $\mathbb{P}^2_*$ (see Proposition 5.11), hence the main results of the paper (Theorems 5.3, 5.4, 5.5 and 5.6 below) are correct. However, as Proposition 5.12 shows, the lemma needs not hold on arbitrary surfaces, and in particular may fail when $S$ is a proper subset of $\mathbb{R}^2$ or $\mathbb{P}^2_*$. Similarly, since [43, Proposition 4.9] is strongly based on it, it works in $\mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{P}^2$ and $\mathbb{P}^2_*$ but not on arbitrary subsurfaces of $\mathbb{S}^2$ and $\mathbb{P}^2$. As a consequence, Theorems 7.1 and 7.2 in [43] are not correct.

In Section 5.1, we will explain why [43, Lemma 4.6] cannot work in general surfaces (we will give some counterexamples to this result in the sphere minus two points and the torus) and we will prove the property for the case of flows defined on $\mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{P}^2$ or $\mathbb{P}^2_*$. On the other hand, we will also point out why there is no way to fix the proof for the characterizations for general open subsets of these three surfaces. In fact, the abovementioned counterexamples provide some $\omega$-limit sets (on the respective open subsets of those surfaces) which are not included in the characterization given in [43] for the cases of proper open subsets of the sphere and the projective plane. Nevertheless, we have found good topological restrictions which a subset of an open set of the sphere must verify in order to be a limit set for an analytic flow on that open set. We present these restrictions in Section 5.2; furthermore, and based on those restrictions, we conjecture a complete topological characterization for such limit sets. We do believe that we will be able to complete a proof for that conjecture in the short term (the solution for the analogous problem in the case of the projective plane is still to be found, but it should follow, with the logical changes, from the previous one).

In order to give formal statements for the characterizations of the limit sets for analytic flows on the whole plane, sphere and projective plane, we must first start by defining several topological notions whose proper combinations allow us to describe those limit sets.

**Definition 5.1.** (see [43, p. 680]) A topological space $A$ is said to be a *cactus* if it is a simply connected union of finitely many disks (when notice that each pair of these disks can have at most one common point). We say that $A \subset \mathbb{R}^2$ is a *half-plane* if both $A$ and $\mathbb{R}^2 \setminus \operatorname{Int} A$ are homeomorphic to $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. We say that $A \subset \mathbb{R}^2$ is a *chain* if there are disks $\{D_i\}_{i\in\mathbb{N}}$ such that:

**(i)** $A = \bigcup_{i\in\mathbb{N}} D_i$;

**(ii)** if $|i - j| = 1$, then $D_i \cap D_j$ consists of exactly one point; otherwise $D_i \cap D_j = \emptyset$;

**(iii)** every bounded set of $\mathbb{R}^2$ intersects finitely many disks $D_i$.

**Definition 5.2.** (see [43, p. 682]) We say that a topological space $A$ is a *bracelet* if it is homeomorphic to $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, (x - 1/2)^2 + y^2 \geq 1/4\}$. We say that $A$ is a *wristlet* if there are finitely many sets $\{B_i\}_{i=1}^n$ (where $n \geq 2$ and every $B_i$ is either a disk or an arc) such that:

**(i)** $A = \bigcup_{i=1}^n B_i$;

**(ii)** if $|i - j| = 1$ or {i,j}={1,n}, then $B_i \cap B_j$ consists of exactly one point; otherwise $B_i \cap B_j = \emptyset$;

**(iii)** every bounded set of $\mathbb{R}^2$ intersects finitely many disks $D_i$.

The announced characterizations for the plane, the sphere and the projective plane are the following ones.

**Theorem 5.3.** *(see [43, The $\mathbb{R}^2$-analytic theorem, pp. 680–681]) Let $\Phi$ be an analytic local flow on the plane and let $\Omega = \omega_\Phi(u)$ for some $u \in \mathbb{R}^2$. Then $\Omega = \operatorname{Bd} A$, with $A$ being*

**(a)** *the empty set;*

**(b)** *a single point;*

**(c)** *a cactus;*

**(d)** *the union of a circle $C$ and finitely many pairwise disjoint cactuses, each of them contained in the disk enclosed by $C$ and intersecting $C$ at exactly one point;*

**(e)** *a union of countably many cactuses, half-planes and chains, which are pairwise disjoint except that each cactus intersects either one of the half-planes or one of the chains at exactly one point; moreover, every bounded set of $\mathbb{R}^2$ intersects finitely many of these sets.*

*Conversely, for every set $A \subset \mathbb{R}^2$ as in (a)–(e) and $\Omega = \mathrm{Bd}\, A$, there are an analytic local flow $\Phi$ on $\mathbb{R}^2$ and a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $h(\Omega)$ is the $\omega$-limit set for some orbit of $\Phi$.*

**Theorem 5.4.** *(see [43, The $\mathbb{R}^2$-polynomial theorem, pp. 681–682]) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be polynomial, let $\Phi$ be the local flow assocaited to $f$, and let $\Omega = \omega_\Phi(u)$ for some $u \in \mathbb{R}^2$. Then $\Omega = \mathrm{Bd}(A)$, with $A$ as in Theorem 5.3, except that in (e) the union is finite and contains no chains.*

*Conversely, for every set $A \subset \mathbb{R}^2$ as before and $\Omega = \mathrm{Bd}(A)$, there are a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(\Omega)$ an $\omega$-limit set for the local flow associated with $f$.*

**Theorem 5.5.** *(see [43, The $\mathbb{S}^2$-analytic theorem, p. 682]) Let $\Phi$ be an analytic local flow on the $\mathbb{S}^2$ and let $\Omega = \omega_\Phi(u)$ for some $u \in \mathbb{S}^2$. Then $\Omega = \mathrm{Bd}\, A$, with $A$ being a single point or a cactus.*

*Conversely, for every such set $A \subset \mathbb{S}^2$, there are an analytic flow $\Phi$ on $\mathbb{S}^2$ and a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $h(\mathrm{Bd}\, A)$ is the $\omega$-limit set for some orbit of $\Phi$.*

**Theorem 5.6.** *(see [43, The $\mathbb{P}^2$-analytic theorem, pp. 680–681]) Let $\Phi$ be an analytic local flow on $\mathbb{P}^2$ and let $\Omega = \omega_\Phi(u)$ for some $u \in \mathbb{R}^2$. Then $\Omega = \mathrm{Bd}\, A$, with $A$ being*

**(a)** *a single point;*

**(b)** *a cactus;*

**(c)** *the union of $M$ (where $M$ is either a nonorientable bracelet. or a nonorientable wristlet, or a Möbius band), with finitely many pairwise disjoint cactuses, each of them intersecting $M$ at exactly one point.*

*Conversely, for every set $A \subset \mathbb{R}^2$ as in (a)–(c) and $\Omega = \mathrm{Bd}\, A$, there are an analytic local flow $\Phi$ on $\mathbb{P}^2$ and a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(\Omega)$ is the $\omega$-limit set for some orbit of $\Phi$.*

## 5.1 A first approximation to the problem

To clarify where the problem in [45] exactly lies, let us state here the content of [43, Lemma 4.6]. We will do it separating the information in that lemma in a theorem and a remark. In Theorem 5.7 we present the part of [43, Lemma 4.6] which is totally correct (the proof given in [43] is completely fine) while in Remark 5.8 we highlight the part which needs not be true.

**Theorem 5.7.** *(c.f. Lemma 4.6 in [43]) Let $S$ be an analytic surface and let $\Phi$ be an analytic local flow on $S$. Let $\Gamma$ be the orbit of $\Phi$ through $u \in S$, put $\Omega = \omega_\Phi(u)$ and assume that $\Omega$ does not consists of a single singular point. Then, for every $v \in \Omega$, there are a disk $D$ neighbouring $v$ and an $n$-star $R \subset \Omega \cap D$, $n \geq 2$, with the following properties:*

**(i)** *$v$ is the center of $R$;*

**(ii)** *the branches of $R$ approach $v$ from definite directions;*

**(iii)** *$R$ intersects $\mathrm{Bd}(D)$ exactly at its endpoints;*

**(iv)** *if $O$ is any of the components of $\mathrm{Int}(D \setminus R)$, then either $O \cap \Gamma$, or $O \cup B$ is a semi-open flow box with border $B$ the boundary of $O$ in $\mathrm{Int}(D)$.*

*Remark* 5.8. In the statement of Lemma 4.6 [43] the following final property is added.

> "Moreover, except for a discrete set of points $P$, the corresponding set $R$ is a 2-star (an arc) and either one of the components of $\mathrm{Int}(D \setminus R)$ does not intersect $\varphi(p)$, or both components, together with its common boundary $B$ in $\mathrm{Int}(D)$ (the arc $R$ minus its both endpoints), are semi flow boxes equally oriented by the flow."

As we will show below (see Proposition 5.11), this property needs not hold for general surfaces.

*Remark* 5.9. When $S$ is a subsurface of the sphere or the projective plane, in Theorem 5.7, we can assume that not only $R \subset \Omega \cap D$ but also $R = \Omega \cap D$ (see [43, p. 690]).

Let us consider a surface $S$ and let $\Phi$ be a (local) flow on $S$. Given a point $u \in S$, we say that $\varphi(u)$ *spirals around* an open arc $B$ of singular points of $\Phi$ if there is an embedding $h : (-1, 1) \times (-1, 1) \to S$ such that:

**(a)** $h((-1, 1) \times \{c\})$ is a semiorbit of $\Phi$ for any $c \neq 0$;

**(b)** $h((-1, 1) \times \{0\}) = B$;

**(c)** semiorbits at both sides of $B$ are oppositely oriented by the flow;

**(d)** $\varphi(u)$ accumulates at $B$, as time goes to $\infty$, from both sides of $B$; more precisely, there is a sequence $t_n \to \infty$ such that the points $\Phi_u(t_n)$ belong to $h(\{0\} \times (-1, 0))$ (respectively, $h(\{0\} \times (0, 1))$) whenever $n$ is odd (respectively, even), and $\Phi_u(t_n) \to h(0, 0)$ as $n \to \infty$.

The problematic part of Lemma 4.6 in [43] is the statement quoted in Remark 5.8, which essentially amounts to say that the "spiralling around" phenomenon cannot occur. The original proof in [43] misses the point that, after cancelling a common analytic factor in a neighbourhood of an arc of singular points, orientations in the resultant vector field can be partially reversed. We next provide a correct, alternative proof for the special cases of the sphere, the plane, the projective plane and the projective plane minus one point. It is based on two parity tricks. The first one is nothing else but the parity of the branches in the local structure of the set of zeros of analytic planar maps (see Theorem A). To present the second one, we first need to introduce an auxiliary concept.

In Chapter 2, we introduced the notion of generalized graph. We say that a compact connected metric generalized graph $G$ is a *graph*. The points of $G$ which are star points of order different to 2 are said to be the *vertexes* of $G$. The set of all vertexes of $G$, $V_G$, is clearly finite; also, the set $G \setminus V_G$ consists of finitely many components: we call any of those components an *edge* of $G$. The following elementary property holds:

**Lemma 5.10.** *Let $G$ be a graph, let $V = \{v_1, v_2, \ldots, v_m\}$ be its set of vertexes and, for every $i \in \{1, 2, \ldots, m\}$, denote by $r_i$ the number of edges of $G$ ending at $v_i$. Then $\sum_{i=1}^m r_i$ is even.*

*Proof.* This follows immediately from the fact that $\sum_{i=1}^m r_i = 2l$, with $l$ the number of edges of the graph. □

**Proposition 5.11.** *Let $\Phi$ be an analytic flow on $S = \mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{P}^2$ or $\mathbb{P}^2_*$. Then $\Phi$ admits no orbits spiralling around open arcs of singular points of $\Phi$.*

*Proof.* We argue to a contradiction by assuming that the orbit of $\Phi$ through a point $u \in S$, $\Gamma_u$, spirals around an open arc $B$ of singular points of $\Phi$, with $h$ being the corresponding embedding. Let $0 < t_0 < s_0$ be such that $\Phi_u(t_0) = h(0, c_0)$, $\Phi_u(s_0) = h(0, d_0)$, with $c_0 < 0 < d_0$. Clearly, there is no loss of generality in assuming that the semiorbit $\Phi_u((t_0, s_0))$ does not intersect $h(\{0\} \times (-1, 1))$, hence $C = \Phi_u([t_0, s_0]) \cup h(\{0\} \times [c_0, d_0])$ is a circle. If $S = \mathbb{P}^2$, then we can additionally assume that $C$ is orientable. Indeed, construct similarly the circle $C' = \Phi([t_1, s_1]) \cup h(\{0\} \times [c_1, d_1])$, with $s_0 < t_1 < s_1$ and $c_1 < 0 < d_1$ such that $\Phi_u(t_1) = h(0, c_1)$ and $\Phi_u(s_1) = h(0, d_1)$. Due to [43, Lemma 4.8], it is not restrictive to suppose that $\omega_\Phi(u)$ intersects $\mathrm{Cl}\, h(\{0\} \times (-1, 1))$ exactly at $h(0, 0)$, and hence that $t_1$ is large enough so that $\Phi_u([t_1, \infty))$ only intersects $h(\{0\} \times (-1, 1))$ at $h(\{0\} \times (c_0, d_0))$, that is, $c_0 < c_1 < d_1 < d_0$. Now realize that (because orientations are reversed at both sides of $B$) $C'$ can be slightly deformed to a homotopic curve $C''$ which does not intersect $C$, and recall that in $\mathbb{P}^2$ any pair of nonorientable circles must have at least one common point (Remark 1.23). Therefore, either $C$ or $C'$ is orientable.

From the previous discussion we conclude that, regardless $S = \mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{P}^2$ or $\mathbb{P}^2_*$, there is a disk or a Möbius band $D \subset S$ such that $C = \mathrm{Bd}\, D$. Let $K$ denote the set of singular points of $\Phi$. According to Theorem A, $K$ is locally a $2n$-star at any of its points, which implies that the component $G$ of $K \cap D$ containing $h(0,0)$ is a graph. Now observe that $G$ is locally a star with an even number of branches at any of its vertexes except $h(0,0)$, where $G$ is locally a 1-star. This contradicts Lemma 5.10. $\square$

In contrast to this, spiralling around is possible, for instance, in the sphere minus two points:

**Proposition 5.12.** *There is an analytic flow on $\mathbb{S}^2 \setminus \{p_N, p_S\}$, where $p_N = (0,0,1)$ and $p_S = (0,0,-1)$, having $(1,0,0)$ and the open arc $B = \{(-\sqrt{1-z^2}, 0, t) : z \in (-1,1)\}$ as its set of singular points, such that all regular orbits spiral around $B$, have $B$ as their $\omega$-limit set, and have $(1,0,0)$ as their $\alpha$-limit set.*

*Proof.* Consider the analytic vector field $g(x,y) = (g_1(x,y), g_2(x,y))$ (on the whole plane) given by

$$\begin{aligned} g_1(x,y) &= \cos(\nu(x,y))\sin y + \sin(\nu(x,y))\sin x, \\ g_2(x,y) &= \sin(\nu(x,y))\sin y - \cos(\nu(x,y))\sin x, \end{aligned}$$

with $\nu(x,y) = \sin^2(x+y)\sin^2(x-y)$. The dynamics of the flow $\Psi$ associated with $g$ in the square $Q = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq \pi\}$ are easy to describe. To begin with, the boundary of $Q$ is invariant under $\Psi$: the only singular points of $g$ in $Q$ are the origin and the four vertexes of the square, and its four edges $\Gamma_1 = \{(x,y) : x+y = \pi,\ 0 < x < \pi\}$, $\Gamma_2 = \{(x,y) : x-y = \pi,\ 0 < x < \pi\}$, $\Gamma_3 = \{(x,y) : x+y = -\pi,\ -\pi < x < 0\}$ and $\Gamma_4 = \{(x,y) : x-y = -\pi,\ -\pi < x < 0\}$ are regular orbits clockwise oriented by the flow. Next observe that the scalar map $V(x,y) = \cos x + \cos y = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$ is nonnegative in $Q$, vanishes at $\operatorname{Bd} Q$ and attains its maximum at the origin. Moreover, $\dot{V}(x,y) = \nabla V(x,y) \cdot g(x,y) = -\sin(\nu(x,y))(\sin^2 x + \sin^2 y) \leq 0$, hence $V$ is a Lyapunov function for $g$, which is strict for all points from $Q$ except those in $\operatorname{Bd} Q$ and in the lines $x+y=0$, $x-y=0$. As it is easy to check, the level curves of $V$ in $Q$ are circles enclosing the origin and all four vertexes of $Q$ are saddle points for $\psi$ (see, e.g., [68, pp. 119–129]; by a *saddle point* we mean a singular point whose every small enough neighbourhoods decompose on four hyperbolic sectors). All these previous fact, together with some elementary Poincaré-Bendixson theory (Theorems 1.53 and 1.54) allows us to conclude that all orbits starting from points in $\operatorname{Int} Q \setminus \{(0,0)\}$ have $(0,0)$ as their $\alpha$-limit set and spiral clockwise towards $\operatorname{Bd} Q$, which is their $\omega$-limit set — see Figure 5.1.

After replacing $g$ by the vector field $w(x,y) = \cos(\frac{x+y}{2})g(x,y)$ we get a new flow having exactly the same orbits and orientations as $\Psi$ in $\operatorname{Int} Q$ (velocities may change); however, orbits $\Gamma_1$ and $\Gamma_3$ of $\psi$ become open arcs of singular points for the new flow. Moreover, for every $k \in \mathbb{Z}$ and every $(x,y) \in Q$, $w(x,y) = w(x+k\pi, y+k\pi)$. Thus, after identifying $\Gamma_1$ and $\Gamma_3$ in $\operatorname{Int} Q \cup \Gamma_1 \cup \Gamma_3$, we obtain an analytic flow on a cylinder.

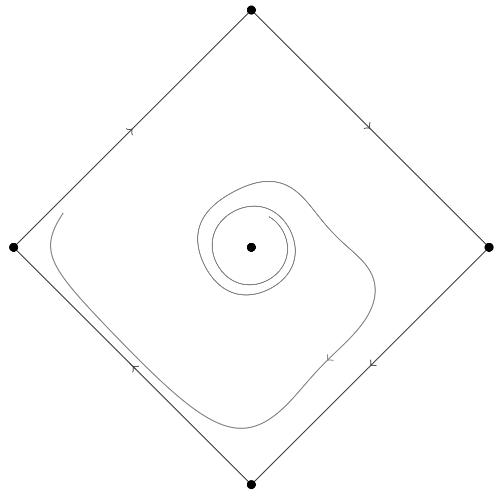

Figure 5.1: Phase portrait of $\Psi$ on $\mathrm{Cl}\, U$.

This last flow can be seen as an analytic flow on the open and bounded euclidean cylinder $L = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, |z| < 1\}$ just applying the diffeomorphism $(x, y) \mapsto (\cos(x+y), \sin(x+y), \frac{y-x}{\pi})$ to the previous one. Realize that this flow has $(1, 0, 0)$ and the open arc $\{(-1, 0, z) : -1 < z < 1\}$ as its set of singular points, and that all regular orbits spiral around that open arc. After projecting horizontally each point of the cylinder $L$ onto the sphere $\mathbb{S}^2$ minus the north and south poles $p_N, p_S$ (this is the so-called Lambert cylindrical projection) we get the desired flow. $\square$

*Remark* 5.13. Since the closure in $\mathbb{S}^2$ of the open arc $B$ above is a (closed) arc, we get a counterexample to the direct statement in [43, Theorem 7.1].

More generally, let $\Phi$ be the analytic flow on $\mathbb{S}^2 \setminus \{p_N, p_S\}$ from Proposition 5.12, $f$ be its associated vector field and $u$ be a regular point for $\Phi$. If $T$ is a totally disconnected compact subset of the orbit $\varphi(u)$, then, after removing the points from $T$, we trivially get an analytic vector field (and hence an analytic flow) on $\mathbb{S}^2 \setminus (\{p_N, p_S\} \cup T)$, similarly having $B$ as an $\omega$-limit set and having orbits spiralling around $B$. Using the topological characterization of noncompact surfaces (see Corollary 1.34) and the well-known fact that two homeomorphic surfaces are analytically diffeomorphic (see Remark 1.17), we conclude that any open subsurface of the sphere minus two points admits analytic flows for which the "spiralling around" phenomenon takes place.

*Remark* 5.14. If $f$ is the analytic vector field associated with the flow on $\mathbb{S}^2 \setminus \{p_N, p_S\}$ from Proposition 5.12, then one can find, as an application of Theorem 1.46, an analytic map $\rho : \mathbb{S}^2 \setminus \{p_N, p_S\} \to (0, \infty)$ with the property that $\rho f$ is $C^\infty$-extensible to the whole sphere just defining it as zero both at $p_N$ and $p_S$. Thus we get a $C^\infty$ flow on the whole sphere having the arc of singular points $B \cup \{p_N, p_S\}$ as its only nontrivial $\omega$-limit set. This, however, does not contradicts [43, Theorem C] (Theorem 5.5 aboce) because analyticity is lost at the endpoints of the arc.

*Remark* 5.15. In order to construct a counterexample as that of Proposition 5.12 one could alternatively start from an analytic scalar map in $\mathbb{S}^2 \setminus \{p_N, p_S\}$ vanishing exactly at $B$, for instance

$$F(x,y,z) = \left(x + \sqrt{x^2+y^2}\right)^2 + y^2,$$

and then apply similar techniques to those in [43, Sections 6 and 7] — c.f. Section 5.2.3 below. This, however, would unnecessarily complicate matters; thus we have preferred a direct, elementary proof in this preliminary section.

On the other hand, these more sophisticated techniques in [43] also allow us to construct a counterexample for the projective plane minus two points. The starting point would be now the map

$$F(x,y,z) = z\left(\left(1 - 2x^2 + \sqrt{(1-2x^2)^2 + y^2}\right)^2 + y^2\right)$$

which, as can be easily seen, is analytic in

$$S = \mathbb{S}^2 \setminus \{(a,0,a),(a,0,-a),(-a,0,a),(-a,0,-a)\},$$

$a = \sqrt{2}/2$, and vanishes at the union $A$ of the equator $\{(x,y,z) \in \mathbb{S}^2 : z = 0\}$ and the two open arcs $\{(x,y,z) \in \mathbb{S}^2 : y = 0 \wedge x \geq \sqrt{2}/2\}$ and $\{(x,y,z) \in \mathbb{S}^2 : y = 0 \wedge x \leq -\sqrt{2}/2\}$. Moreover, we have $F(-x,-y,-z) = -F(x,y,z)$. Then, reasoning as in [43, Sections 6 and 7], one gets an analytic vector field $f$ on $S$, also satisfying $f(-x,-y,-z) = -f(x,y,z)$. The *Hartman-Grobman Theorem* (see, e.g., [68, pp. 119–129]) guarantees that the poles are repelling foci (by a *repelling focus* we mean an isolated singular point $q$ which possesses a neighbourhood where all points have $\{q\}$ as $\alpha$-limit set); moreover, $f$ has regular orbits converging towards $A$ (more precisely, those at the north hemisphere $\mathbb{S}^2_+ = \{(x,y,z) \in \mathbb{S}^2 : z \geq 0\}$ have $\mathbb{S}^2_+ \cap A$ as their $\omega$-limit set, and those at the south hemisphere $\mathbb{S}^2_- = \{(x,y,z) \in \mathbb{S}^2 : z \leq 0\}$ have $\mathbb{S}^2_- \cap A$ as their $\omega$-limit set). After canonically identifying opposite points in the sphere, we obtain an analytic flow on the projective plane minus two points (this is the reason why we need $f(-x,-y,-z) = -f(x,y,z)$) having the union of a nonorientable circle $C$ and an open arc $B$ (intersecting transversally at exactly one point $p$) as an $\omega$-limit set. This is in disagreement with the direct statement in [43, Theorem 7.2]. As it turns out, there are regular orbits of this flow which spiral around the two open arcs into which $p$ decomposes $B$.

*Remark* 5.16. If in Proposition 5.12 we use the vector field

$$\tilde{w_1}(x,y) = \cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) g_1(x,y)$$

instead of $w_1$, then we get $\tilde{w_1}(x,y) = \tilde{w_1}(x+k\pi, y+l\pi)$ for all integers $k$ and $l$, so this vector field induces an analytic flow on the torus having the union $\Omega = C_1 \cup C_2$ of two non-null homotopic transversal circles of singular points as an $\omega$-limit set; moreover, any orbit having $\Omega$ as its $\omega$-limit set spirals around all open arcs in $\Omega$ not containing the intersection point of $C_1$ and $C_2$. Hence the statement from [43] highlighted in Remark 5.8 may fail even when the surface $S$ is compact.

## 5.2 A topological characterization of $\omega$-limit sets for analytic flows on open subsets of the sphere

We devote this section to tackle the problem of characterizing (topologically) the $\omega$-limit sets for analytic flows on open subsets of the sphere.

### 5.2.1 Introductory notions and statement of the main results

Throughout this chapter, the real euclidean distance $d(\cdot,\cdot)$ in the unit sphere $\mathbb{S}^2 = \{(u_1,u_2,u_3) \in \mathbb{R}^3 : u_1^2+u_2^2+u_3^2 = 1\}$ will remain fixed.

Recall that a Peano space is a locally connected metric continuum. Single points, arcs, circles and disks are the simplest examples of Peano spaces. Throughout the rest of the chapter, some other special types of Peano spaces will be extensively used. We describe them below.

We say that a Peano space $X$ is a *net* when the set $E_X$ of all points admitting an open arc as a neighbourhood is dense in $X$. Each component of $E_X$ is called an *edge* of $X$, and the points of $V_X = X \setminus E_X$ are called the *vertexes* of $X$. As it turns out, any edge $E$ of $X$ is either a circle (when $E = X$) or an open arc. In this last case $\operatorname{Cl} E$ is either $E$ plus one vertex of $X$ (and then we get a circle) or $E$ plus two vertexes of $X$ (and then we get an arc).

*Remark* 5.17. The previous statements can be proved as follows. Let $E$ be an edge of $X$, fix $x \in E$, find open arcs $A, B$ in $E$ neighbouring $x$ with $\operatorname{Cl} A \subset B$ and let $p$ and $q$ be the endpoints of $\operatorname{Cl} A$. Then $X \setminus A$ is trivially locally connected. Indeed, assume first that this set is connected, hence a Peano space. Then there is an arc $C \subset X \setminus A$ with endpoints $p$ and $q$. If $C \subset E$, then, by connectedness, $E$ equals the circle $C \cup A$. Otherwise, there is a arc $C_p \subset C$ with endpoints $p$ and $v$, and an arc $C_q \subset C$ with endpoints $q$ and $w$, such that $v$ and $w$ are vertexes of $X$ and both $C_p \setminus \{v\}$ and $C_q \setminus \{w\}$ are included in $E$. Again using the connectedness of $E$, if $v = w$ then $E \cup \{v\}$ is the circle $C \cup A$, while if $v \neq w$ then $E \cup \{v,w\}$ is the arc $A \cup C_p \cup C_q$. On the other hand, if $X \setminus A$ is not connected, then it is the union of two disjoint Peano spaces $V \ni p$ and $W \ni q$. We claim that $V$ (and similarly $W$) is not fully included in $E$. If this is not true, then any point of $V$ except

$p$ disconnects $V$ (otherwise we could argue as in the above paragraph to find a circle in $V \subsetneqq E$, then arriving at a contradiction), hence any pair of points of $V$ disconnect $V$. By [54, Theorem 2, p. 180], $V$ is then a circle and again we get a contradiction. Thus, there are points in $V$ which do not belong to $E$, and we can construct an arc $C_p$ with endpoints $p$ and a vertex $v$ in $V$, such that $C_p \setminus \{v\} \subset E$. Arguing similarly in $W$ to find a vertex $w \in W$ and an arc $C_q$ with endpoints $q$ and $w$ and such that $C_w \setminus \{w\} \subset E$, we conclude as before that $E \cup \{v, w\}$ is the arc $A \cup C_p \cup C_q$.

A graph is then a net with finitely many vertexes and edges. If a graph includes no circles, then it is called a *tree*; more generally, a Peano space including no circles is called a *dendrite*. We have already remarked that any Peano space is arcwise connected. For dendrites a stronger property holds: any two different points $p$ and $q$ in a dendrite $X$ are the endpoints of a unique arc $X_{p,q}$ in $X$. Accordingly, the locally arcwise connectedness of a dendrite $X$ can be rewritten as follows: for every $p \in X$ and every $\epsilon > 0$, there exists $\delta > 0$ such that if $q$ is a point in the open ball of centre $p$ and radius $\delta$, the diameter of $X_{p,q}$ is less than $\epsilon$.

If a tree $X$ has $n$ edges, then it has $n+1$ vertexes: if, moreover, there is a vertex $c$ belonging to the closure of all its edges, then $X$ is nothing else but an $n$-star with center $c$ and endpoints all other vertexes of $X$.

The next one is the most important notion of this paper.

**Definition 5.18.** We say that $\emptyset \subsetneqq A \subsetneqq \mathbb{S}^2$ is a *shrub* if it is a simply connected Peano space.

*Remark* 5.19. If $A$ is a shrub, then all components $\{R_j\}_j$ of $\operatorname{Int} A$ are open disks (because $R$ is connected, hence $\mathbb{S}^2 \setminus R_j = R \cup \operatorname{Bd} A \cup \bigcup_{j' \neq j} R_{j'}$ is connected as well). If fact, more is true: their closures $D_j = \operatorname{Cl} R_j$ are disks. To prove this it is enough to show, according to [73, Remark 14.20(a), p. 291], that if a sequence $(u_n)_{n=1}^{\infty}$ of points in $R_j$ converges to a point $u \in \operatorname{Bd} R_j$, then there is a path in $D_j$ monotonically passing through the points $u_n$ and ending at $u$ (that this, there is a continuous map $\varphi : [0,1] \to D_j$ and numbers $0 \leq t_1 < t_2 < \cdots$, $t_n \to 1$, such that $\varphi(t_n) = u_n$ for all $n$ and $\varphi([0,1)) \subset R_j$). This last statement is a direct consequence of the following fact: let $\epsilon > 0$, then there is $\delta > 0$ such that, whenever $v, w \in R_j$ and $0 < d(v,w) < \delta$, we can find an arc in $R_j$ with endpoints $v, w$ whose diameter is less than $\epsilon$. Certainly, such a $\delta > 0$ exists, because $A$ is a Peano space [54, Theorem 2, p. 253 and Theorem 1, p. 254], except than we cannot guarantee that the small arc connecting $v$ and $w$, call it $L$, is fully contained in $R_j$. One thing, at least, is sure: $L \subset D_j$. Otherwise, we could easily construct a circle $C \subset A$ intersecting both $R_j$ and $A \setminus R_j$, and simply connectedness forces that one of the open disks enclosed by $C$ is included in $A$, which contradicts that $R_j$ is a component of $\operatorname{Int} A$. Thus, $L \subset D_j$ and, similarly as above, we can construct a circle $C'$ in $D_j$ including all points of $L \cap \operatorname{Bd} R_j$. Since $C'$ encloses an open disk fully included in $D_j$, it is easy to slightly modify $L$ so that the resultant arc $L'$ still has diameter less than $\epsilon$, and $v$ and $w$ as its endpoints, and additionally satisfies $L' \subset R_j$.

We call the disks of the above (countable) family $\{D_j\}_j$ the *leaves* of the shrub $A$. Observe that, again due to the simply connectedness of $A$, any circle in $A$ must be included in one of its leaves. This implies, in particular, that distinct leaves of a shrub can have at most one common point. All dendrites in $\mathbb{S}^2$ are shrubs [54, Theorem 2, p. 375 and Corollary 7, p. 378]; conversely, if a shrub has no leaves, then it is a dendrite. If a shrub $A$ is a union of finitely many leaves, then it is called a *cactus*. If $A$ is the union of a cactus $D$ and finitely many arcs $\{A_i\}_{i=1}^m$, with every $A_i$ intersecting $D$ at only one endpoint of $A_i$, $u_i$, and $(A_i \setminus \{u_i\}) \cap (A_{i'} \setminus \{u_{i'}\}) = \emptyset$ whenever $i \neq i'$, we call $A$ an *$m$-prickly cactus*, with the arcs $A_i$ being the *prickles* of $A$ (here $m = 0$ is possible, when we get just a cactus); the endpoints of the prickles of $A$ not belonging to $D$ will also be called the *endpoints* of $A$. If all points of all circles in $\operatorname{Bd} A$ are star points in $\operatorname{Bd} A$, we say that the shrub $A$ is *simple*. In particular, all cactuses and prickly cactuses are simple shrubs.

**Definition 5.20.** Let $A$ be a shrub.

- Let $u \in \operatorname{Bd} A$. We say that $u$ is an *odd vertex (of $A$)* if either $u$ is not a star point in $\operatorname{Bd} A$ or $u$ is in no leaf of $A$ and, for some odd positive integer $n$, $u$ is a star point in $\operatorname{Bd} A$ of order $n$.
- Let $K$ be a maximal connected union of leaves of $A$. We say that $K$ is an *odd cactus (in $A$)* if there is an $n$-prickly cactus neighbouring $K$ in $A$ for some odd number $n$.

**Definition 5.21.** We say that a shrub $A$ is *realizable* if its set of odd vertexes is totally disconnected.

*Remark* 5.22. If $A$ is a realizable shrub, then $\operatorname{Bd} A$ is a net and all odd vertexes of $A$ are vertexes of $\operatorname{Bd} A$.

*Remark* 5.23. Clearly, the set of odd vertexes of a shrub is closed, and a set consisting of all odd vertexes of a shrub and one point from each of its odd cactuses, is closed (and totally disconnected if the shrub is realizable) as well.

Let $O$ be a nonempty open subset of $\mathbb{S}^2$. If $f$ is an analytic vector field on $O$ we will assume in what follows that $f$ can be $C^\infty$-extended to the whole $\mathbb{S}^2$ by adding singular points at $\mathbb{S}^2 \setminus O$. In view of Theorem 1.46 above, this involves no loss of generality (because after multiplying a vector field by a positive factor the resultant vector field has exactly the same $\omega$-limit sets as the previous one). Hence its associated flow can be seen (and so we will do) as globally defined on $\mathbb{R} \times \mathbb{S}^2$, and when speaking about $\omega$-limit sets for $f$, this is the flow we are referring to.

Recall that we are interested in characterizing, up to homeomorphism, the $\omega$-limit set for analytic flows on $O$. Without loss of generality we may assume that $O$ is a region and, because of Theorem 1.4(i), that $\mathbb{S}^2 \setminus O$ is a totally disconnected set. Now we have:

**Theorem F.** *Let $f$ be an analytic vector field on $O$ and assume that $T = \mathbb{S}^2 \setminus O$ is totally disconnected. Then any $\omega$-limit set for $f$ is the boundary of a shrub. Moreover, all odd vertexes of the shrub are contained in $T$ (hence it is realizable) and every odd cactus in the shrub must intersect $T$.*

We present the proof of this last result in the following subsection. We do believe that these restrictions are in fact the properties which characterize $\omega$-limit sets for analytic flows on open sets of the sphere. A first step to prove the converse of Theorem F is given with the following proposition (whose proof is tackled in Subsection 5.2.3 below), because we conjecture that the boundary of a realizable shrub can be realized, up to a "small" set of points, as an analytic set.

**Proposition 5.24.** *Let $O$ be a simply connected region of $\mathbb{S}^2$, write $\Omega = \mathrm{Bd}\, O$, and let $F : \mathbb{S}^2 \to \mathbb{R}$ be a $C^\infty$ map, which is analytic (at least) in $O$, and satisfies $F(u) \neq 0$ for any $u \in O$ and $F(u) = 0$ for any $u \in \Omega$. Then there is a $C^\infty$ vector field $f$ in $\mathbb{S}^2$, which is analytic wherever $F$ is (in particular, in $O$), and such that its associated flow has $\Omega$ as one of its $\omega$-limit sets.*

To conclude, these two previous results allow us to conjecture the following converse.

**Conjecture 1.** *Let $A \subset \mathbb{S}^2$ be a realizable shrub and let $T \subset A$ contain all odd vertexes of $A$ and one point from each of the odd cactuses of $A$. Then there are a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$, and an analytic vector field on $h(\mathbb{S}^2 \setminus T)$, such that its associated flow has the boundary of $h(A)$ as an $\omega$-limit set.*

And consequently:

**Conjecture 2.** *Up to homeomorphisms, a set is an $\omega$-limit set for some analytic vector field defined on $\mathbb{S}^2$ except for a totally disconnected complementary if and only if it is the boundary of a realizable shrub.*

### 5.2.2 Proof of Theorem F

Let $\Phi$ be the (global) flow associated with $f$. Let $p \in \mathbb{S}^2$ and rewrite $\Gamma = \Phi_p(\mathbb{R})$, $\Omega = \omega_\Phi(p)$. Clearly we can discard the cases when $\Omega$ is a singleton or a circle. Moreover, since no local flow on the sphere admits no nontrivial recurrent orbits, $\Gamma \cap \Omega = \emptyset$.

Also, several "intuitive" (but deep) topological results from the topology of the sphere will be needed. The following ones may not be as well known as the Jordan curve theorem, but they will be implicitly used a number of times: if $V \subset \mathbb{S}^2$ is compact and totally disconnected, then there is an arc in $\mathbb{S}^2$ including $V$ [54, Theorem 5, p. 539 (see also p. 189)]; if $B$ and $B'$ are either arcs or circles in $\mathbb{S}^2$, then there is a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ mapping $B$ onto $B'$ — see Section 1.1.4 (alternatively, see [54, Corollary 2, p. 535]).

**Lemma 5.25.** *$\Omega$ is a net (hence a Peano space), and the same is true for all its subcontinua.*

*Proof.* Due to the compactness of $\mathbb{S}^2$, $\Omega$ is a continuum [13, Theorem 3.6, p. 24]. Essentially it suffices to show that any subcontinuum of $\Omega$ is a Peano space, that is, locally connected, because then Theorem 5.7 easily implies that it is a net. To prove the local connectedness, according to [54, Theorem 2, p. 247], we only need to show that if $K$ is a nowhere dense subcontinuum of $\Omega$, then it is a singleton. (Recall that a nowhere dense subset of a topological space is one whose closure has empty interior). Suppose the opposite to find such a subcontinuum $K$ having at least two points. Since $T$ is totally disconnected, $K$ cannot be included in $\Omega \cap T$ and we can find a point $u \in K \cap O$. Let $X \subset \Omega \cap O$ be a star neighbouring $u$ and having it as its center (Theorem 5.7). Two possibilities arise: either $K$ intersects $X$ exactly at $u$, or $K \cap X$ contains an arc. Both of them are impossible: the first one because of the connectedness of $K$, the second one because $K$ has empty interior in $\Omega$. □

Since we are assuming that $\Omega$ is neither a circle nor a singleton, it is the union of its nonempty families of edges (which are countably many) and vertexes. Let $E$ be an edge of $\Omega$ and $u \in E$. We say that $u$ is *two-sided* if there is a disk $D$ neighbouring $u$ such that $D$ is decomposed by $E$ into two components $D_1$ and $D_2$, and $\Gamma$ accumulates at $u$ from both $D_1$ and $D_2$. Otherwise we say that $u$ is *one-sided.* If $u \in E$ is regular, then there is a flow box $M$ such that $h(0,0) = u$ for the corresponding homeomorphism $h : [-1,1] \times [-1,1] \to M$. Since the arc $h(\{0\} \times [-1,1])$ is transversal to the flow, it must be intersected monotonically, as time increases, by $\Gamma$. Hence $u$ is one-sided.

We say that an edge $E$ is *two-sided* if it has some two-sided point; otherwise, it is called *one-sided.*

**Lemma 5.26.** *Let $E$ be an edge of $\Omega$. Then $E$ is one-sided if and only if it is contained in a circle in $\Omega$. Moreover, if $E$ is two-sided, then all points from $E$ are two-sided.*

*Proof.* The "if" part of the first statement is obvious. Next we prove that if $E$ is not contained in a circle, then it is two-sided.

By Lemma 5.25, there are disjoint continua $\Omega_1, \Omega_2$ satisfying $\Omega \setminus E = \Omega_1 \cup \Omega_2$. Use [54, Theorem 5', p. 513] to find a circle $C \subset \mathbb{S}^2$ separating $\Omega_1$ and $\Omega_2$. Clearly, we can assume that $C$ intersects $E$ (hence $\Omega$) exactly at one point $u \in O$ which, arguing to a contradiction, we will suppose one-sided. By Theorem 5.7, there is a semi-flow box $M$, with corresponding homeomorphism $h : [-1,1] \times [0,1] \to M$ and border $B \subset E$, such that $h(0,0) = u$ and $\Gamma$ accumulates at $B$ from $M$. We can assume, without loss of generality, that $M$ intersects $C$ at the arc $L = h(\{0\} \times [0,1])$. After crossing $L$ the semiorbits of $\Gamma$ in $M$ enter, as time increases, into one of the open disks enclosed by $C$, call it $U$. But then $\Gamma$

must also cross $C$ infinitely many times to escape from $U$, and these other crossing points cannot belong to $M$ (and hence cannot be close to $u$ because $u$ is one-sided). Consequently, $\Omega$, the $\omega$-limit set of $\Gamma$, intersects $C \setminus \{u\}$, and we get the desired contradiction.

The above argument implies in fact that if $E$ is two-sided, then all points from $E \cap O$ are two-sided. Since this set is dense in $E$, all points from $E$ are two-sided. □

Let $\{C_j\}_j$ be the family of circles in $\Omega$. Each $C_j$ decomposes $\mathbb{S}^2$ into open disks $R_j$ and $S_j$, which can be chosen so that the resultant disks $D_j = C_j \cup R_j$ do not intersect $\Gamma$ (hence $R_j$ is a component of $\mathbb{S}^2 \setminus \Omega$ for any $j$). Let $R$ be the component of $\mathbb{S}^2 \setminus \Omega$ containing $\Gamma$. Then the family of components of $\mathbb{S}^2 \setminus \Omega$ is precisely $\{R\} \cup \{R_j\}_j$. Indeed, assume that $U$ is a component of $\mathbb{S}^2 \setminus \Omega$ different from $R$ and any $R_j$. Lemma 5.26 implies that $\operatorname{Bd} U$ can intersect no edge of $\Omega$; therefore, $\operatorname{Bd} U$ is totally disconnected and $W = \mathbb{S}^2 \setminus \operatorname{Bd} U$ is a region. Since $\operatorname{Bd} W = \operatorname{Bd} U$, $U \subset W$ and both $U$ and $W$ are regions, we get $U = W$, which is impossible.

Let $A = \Omega \cup \bigcup_j R_j = \mathbb{S}^2 \setminus R$. We have:

**Lemma 5.27.** *$A$ is a shrub and $\Omega = \operatorname{Bd} A$, the leaves of $A$ being the disks $D_j$.*

*Proof.* Since $\operatorname{Int} \Omega = \emptyset$, we have $\Omega = \operatorname{Bd} A$. Since $R$ is connected, it suffices to show that $A$ is locally connected.

If the family $\{D_j\}_{j=1}^k$ is finite this is simple: just use the Hahn-Mazurkiewicz theorem to find continuous onto maps $\varphi : [0,1] \to \Omega$ (here we use Lemma 5.25), $\varphi_j : [0,1] \to D_j$, and combine these $k+1$ maps to generate a continuous map applying $[0,1]$ onto $A$.

If $\{D_j\}_{j=1}^\infty$ is infinite, then the above argument still works provided that the diameters of the disks $D_j$ tend to zero. Assume that the opposite is true to find $\delta > 0$ and disks $D_{j_n}$ so that $\operatorname{diam} D_{j_n} = d(u_n, v_n) \geq \delta$ for appropriate $u_n, v_n \in D_{j_n}$, $n = 1, 2, \ldots$. We can assume that the sequence $(u_n)$ converges, say to $u \in A$. If $u$ belongs to one of the open disks $R_j$ or to $\Omega \cap O$, then we immediately get a contradiction (recall Theorem 5.7), so $u$ must belong to $T$. Since $T$ is totally disconnected, we can find a disk $D$ neighbouring $u$ as small as needed (in particular, $\operatorname{diam} D < \delta$) so that $\operatorname{Bd} D \subset O$. If $n$ is large enough, then $u_n \in D$ and $v_n \notin D$, hence $D_{j_n}$ intersects $\operatorname{Bd} D$. Thus the disks $D_{j_n}$ accumulate at a point from $O$ and again we get a contradiction. □

We are ready to finish the proof of Theorem F. After Lemma 5.27, we are left to show that all odd vertexes of $A$ are in $T$ and all odd cactuses in $A$ intersect $T$.

Let $P = \operatorname{Sing}(\Phi) \cap O$, assume that $u \in O$ is an odd vertex of $A$ and let $X$ be an $m$-star as in Theorem 5.7. Then $m$ is odd and, since there are no disks $D_j$ near $u$, all edges ending at $u$ must be two-sided (Lemma 5.26). In particular, all points of $X$ must be singular for $\Phi$. Now, since $P$ is the set of zeros of an analytic function $F : O \to \mathbb{R}$, it is locally at $u$ a $2n$-star $Y$ for some nonnegative integer $n$ (Theorem A). Since $X$ is "odd" and $Y$ is "even", $Y$ strictly includes $X$. This means that (because all points from $X \setminus \{u\}$ are two-sided) there is a semi-flow box having two consecutive branches as its border, and intersecting a branch of $Y$ not included in $X$. This is impossible, because all singular points of a semi-flow box belong to its border.

Finally, assume that $K \subset O$ is and odd cactus, when the $m$-prickly cactus $L$ neighbouring $K$ in $A$ can be assumed to be included in $O$ as well. Let $N$ be the set of endpoints of $L$. Again, all edges ending at $K$ are two-sided, hence all prickles of $L$ consist of singular points. Note that there are no singular points outside $L$ accumulating at $L \setminus N$; otherwise there would be an arc in $O$ intersecting $L \setminus N$ at exactly one point, and we could reason to a contradiction with similar arguments to those in the paragraph above. The conclusion is: $G = P \cap L$ is the union of finitely many pairwise disjoint graphs, which are locally "even" at all their vertexes, except for the $m$ endpoints of $L$. This contradicts the following general parity property for graphs: if $V = \{v_1, v_2, \ldots, v_l\}$ is the set of vertexes of a graph $G$ and, for every $i \in \{1, 2, \ldots, l\}$, $r_i$ denotes the order of $v_i$ as a star point in $G$, then $\sum_{i=1}^{l} r_i$ is even (see Lemma 5.10).

### 5.2.3 Proof of Proposition 5.24

In what follows we assume, without loss of generality and after applying appropriate analytic transformations, that the north pole $p_N = (0,0,1)$ of $\mathbb{S}^2$ belongs to $\Omega$, that the south pole $p_S = (0,0,-1)$ belongs to $O$, and that the meridian $I_0$ consisting of the points $(\sqrt{1-z^2}, 0, z)$, $z \in [-1,1]$, is included in $O \cup \{p_N\}$. (More in general, by a *meridian* we mean an arc in $\mathbb{S}^2$ having $p_N$ and $p_S$ as its endpoints and which is included in $O \cup \{p_N\}$.)

As it turns out, the vector field $f$ we are looking for can be explicitly derived from $F$, which immediately guarantees that it satisfies the smoothness requirements from the theorem. Namely, let $\|\cdot\|$ denote the euclidean norm, let $G : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$ be given by $G(u) = F^2(u/\|u\|)$, and define $f : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^3$, $f = (f_1, f_2, f_3)$, as follows:

$$\begin{aligned}
f_1(x,y,z) &= 2z(y-x)G(x,y,z) + (x^2+y^2)\left(-y\frac{\partial G}{\partial z}(x,y,z) + z\frac{\partial G}{\partial y}(x,y,z)\right), \\
f_2(x,y,z) &= -2z(x+y)G(x,y,z) + (x^2+y^2)\left(x\frac{\partial G}{\partial z}(x,y,z) - z\frac{\partial G}{\partial x}(x,y,z)\right), \\
f_3(x,y,z) &= (x^2+y^2)\left(2G(x,y,z) + y\frac{\partial G}{\partial x}(x,y,z) - x\frac{\partial G}{\partial y}(x,y,z)\right).
\end{aligned}$$

It is easy to check that $f(u) \cdot u = 0$ for any $u$. Hence $f$, when restricted to $\mathbb{S}^2$, induces a vector field on $\mathbb{S}^2$. Observe that, because of the definition of $G$, all points of $\Omega$ are singular points for the corresponding system

$$u' = f|_{\mathbb{S}}^2(u). \tag{5.1}$$

On the other hand, although $G$ is positive on $O$, there may be many singular points of (5.1) in $O$ ($p_S$, for instance, is one of them).

Next we will show, through a sequence of lemmas, that $\Omega$ is an $\omega$-limit set for (5.1), but first the point behind the definition of $f$ must be clarified. For this, consider the semi-space $U = \{(x,y,z) \in \mathbb{R}^3 : z < 1\}$ and the map $\pi : U \to \mathbb{R}^2$ given by $\pi(x,y,z) = (x/(1-z), y/(1-z))$, which of course becomes the stereographic projection when restricted to $\mathbb{S}^2$. Recall that if a meridian $I$ is given, then there exists an analytic map $\Lambda_I : \mathbb{R}^2 \setminus \pi(I \setminus \{p_N\}) \to \mathbb{R}$ such that $\Lambda_I(x,y) \in \arg(x+iy)$ for any $(x,y)$. Likewise, let $U_I = U \setminus \pi^{-1}(\pi(I \setminus \{p_N\}))$ and define $\Theta_I : U_I \to \mathbb{R}$ by $\Theta_I = \Lambda_I \circ \pi$. Note that $\Theta_I$ can be locally written as $\Theta_I(x,y,z) = k\pi + \arctan(y/x)$ or $\Theta_I(x,y,z) = k\pi + \operatorname{arccot}(x/y)$ for some integer $k$, and then

$$\nabla\Theta_I(x,y,z) = \left(\frac{\partial\Theta_I}{\partial x}(x,y,z), \frac{\partial\Theta_I}{\partial y}(x,y,z), \frac{\partial\Theta_I}{\partial z}(x,y,z)\right) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right).$$

Finally, write $\rho(x,y,z) = (x^2+y^2)G(x,y,z)$, $J_I(u) = \rho(u)e^{-2\Theta_I(u)}$ and $H_I(u) = \log J_I(u)$. While $J_I$ is well defined in $U_I$, $H_I$ only makes sense in the open set $V_I = U_I \cap G^{-1}((0,\infty))$. Still, observe that $O \setminus I \subset V_I$.

Fix a meridian $I$. The key property of $f$ is that, as it can be easily checked, we can write it as

$$f(u) = \rho(u)(\nabla H_I(u) \times u) \text{ whenever } u \in V_I.$$

This has the important consequence that $\nabla H_I(u) \cdot u' = 0$ for some relevant (connected) smooth curves $u(t) = (x(t), y(t), z(t))$ in $O \setminus I$, which means that $H_I$ (and consequently $J_I$) is constant on them. Such is the case, for instance, if $u(t)$ is a solution of the system (5.1), because then

$$\nabla H_I(u) \cdot u' = \rho(u)\nabla H_I(u) \cdot (\nabla H_I(u) \times u) = 0,$$

and also if all points of the curve $u(t)$ are singular, because then $\nabla H_I(u) \times u = 0$, which implies that $\nabla H_I(u) = \kappa(u)u$ for some scalar map $\kappa$, and therefore

$$\nabla H_I(u) \cdot u' = \kappa(u)(u \cdot u') = 0,$$

the last equality just following from the fact that $u(t)$ is a curve in the sphere $\mathbb{S}^2$.

The above properties can be exploited further. Firstly, Theorem A implies that if $\Phi$ is the flow associated with (5.1), then $J_I$ is in fact locally constant in $\operatorname{Sing}(\Phi) \cap (O \setminus I)$, hence constant on each of the components of this set. On the other hand, if $p \in O \setminus \{p_S\}$, then we cannot automatically guarantee the constancy of some concrete map $J_I$ on the whole maximal solution $\Phi_p(t) = (x_p(t), y_p(t), z_p(t))$, because although the orbit lies in $O$ it needs not be fully included in any region $O \setminus I$. Still, it is clearly possible to find a continuous choice of the angle (that is a continuous map $\theta_p : \mathbb{R} \to \mathbb{R}$ satisfying $\theta_p(t) \in \arg(x_p(t) + iy_p(t))$ for any $t \in \mathbb{R}$), so that

$$w_p(t) = \rho(\Phi_p(t))e^{-2\theta_p(t)}$$

is constant. We gather these results as a lemma:

**Lemma 5.28.** *The following statement holds:*

*(i) If $I$ is a meridian, then $J_I$ is constant on any component of* $\operatorname{Sing}(\Phi) \cap (O \setminus I)$.

*(ii) If $p \in O \setminus \{p_S\}$ and $I$ is a meridian, then $J_I$ is constant on every semiorbit of $\Phi_p(\mathbb{R})$ included in $O \setminus I$; moveover, the above map $w_p(t)$ is constant.*

**Lemma 5.29.** *The south pole $p_S$ is a repelling focus for $\Phi$.*

*Proof.* It suffices to show that the same statement is true, with respect to the origin when we transport the system (5.1) to $\mathbb{R}^2$ via the local chart $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$, when we obtain the system given by

$$g(x, y) = (f_1(x, y, \sqrt{1 - x^2 - y^2}), f_2(x, y, \sqrt{1 - x^2 - y^2})).$$

Now a direct calculation shows that the jacobian matriz of $g$ at $(0, 0)$ is

$$Jg(0, 0) = \begin{bmatrix} 2G(p_S) & -2G(p_S) \\ 2G(p_S) & 2G(p_S) \end{bmatrix},$$

its eigenvalues being $2G(p_S)(1 \pm i)$. Since $G(p_S) > 0$, the lemma follows (because of the Hartman-Grobman Theorem, c.f. [68, pp. 119–129]). □

**Lemma 5.30.** *Let $P$ be the set of singular points in $O$ which are nontrivial $\omega$-limit sets (that is, $p \in P$ if and only if there is $q \neq p$ such that $\omega_\Phi(q) = \{p\}$). Then, for any $p \in P$, there are only finitely many orbits having $p$ as its $\omega$-limit set. Moreover, $P$ is discrete, that is, all its points are isolated, hence countable.*

*Proof.* Let $p \in P$, fix a meridian $I$ not containing $p$ and say $J_I(p) = a$. Find a small star $X \subset O \setminus I$ neighbouring $p$ in $\operatorname{Sing} \Phi$ (recall that $X$ becomes to a 0-star, that is, just the point $p$, when $p$ is isolated in $\operatorname{Sing} \Phi$). Now realize that, by Lemma 5.28 and continuity, $J_I$ also equals $a$ on $X$ and all small semiorbits ending at points from $X$. Since $J_I^{-1}(\{a\})$ is analytic, and $J_I$ cannot be constant on $O \setminus I$ (because then $f$ would vanish on the whole $O$, which is not true in view of Lemma 5.29), the lemma follows immediately from Theorem A. □

**Lemma 5.31.** *Let $p \in O$ and assume that $\alpha_\Phi(p) = p_S$ and $\omega_\Phi(p)$ is not a singular point of $O$. Then $\omega_\Phi(p) = \Omega$.*

*Proof.* Rewrite $u(t) = \Phi_p(t)$, $\Gamma = \Phi_p(\mathbb{R})$, $\theta(t) = \theta_p(t)$, $w(t) = w_p(t)$, $\Omega' = \omega_\Phi(p)$.

First we show that $\Omega' \subset \Omega$. Suppose not to find $q \in \Omega' \cap O$ and (using Theorem 5.7) a semi-flow box $M \subset O$ whose border $B$ is in $\Omega'$ and such that $\Gamma$ accumulates at $B$ from $M \setminus B$. If $h : [-1,1] \times [0,1] \to M$ is the corresponding homeomorphism, then the points $q_n = u(t_n)$ $(t_n \geq 0)$ of $\Gamma$ intersecting $A = h(\{0\} \times [0,1])$ converge monotonically, as time increases, to $q$. If $A_n$ is the arc in $A$ with endpoints $q_n$ and $q_{n+1}$, then two possibilities arise: either one of the circles $C_n = A_n \cup u([t_n, t_{n+1}])$ separates $q$ and $p_N$, or neither of them does.

Assume that the first possibility holds. If, say, $C_{n_0}$ separates $q$ and $p_N$, then there is a meridian $I$ such that neither $u([t_{n_0}, \infty))$ nor $\Omega'$ intersect it (we are using $\alpha_\Phi(p) = p_S$). Hence $J_I$ equals to a constant $a$ on $u([t_{n_0}, \infty))$ (Lemma 5.28(ii)). Moreover, by continuity, $J_{\Theta_I} = a$ on $\Omega'$ as well. Now, since $J_I^{-1}(\{a\})$ is an analytic set which it is not locally a star at $q$, we get $J_I^{-1}(\{a\}) = O \setminus I$ (Theorem A), which is impossible.

If the second possibility holds, then all curves $C_n$ have the same winding number $\nu \in \{-1, 1\}$ around $p_N$, and $\theta(t_{n+1}) - \theta(t_n) \to 2\pi\nu$ as $n \to \infty$. Therefore, $|\theta(t_n)| \to \infty$. Since $\rho$ is positive in $q$, it is impossible that $w(t)$ is constant, contradicting Lemma 5.28(ii). This concludes the proof that $\Omega' \subset \Omega$.

We are now ready to prove $\Omega' = \Omega$. Note firstly that, since $d(u(t), \Omega) \to 0$ as $t \to \infty$ (because $\Omega' \subset \Omega$) and $G$ vanishes at $\Omega$, the only way for Lemma 5.28(ii) to hold is that $\theta(t) \to -\infty$ as $t \to \infty$. We can, of course, assume $\theta(0) \geq 0$, hence the last and first numbers $t_n$ and $s_n$ respectively satisfying $\theta(t_n) = -2\pi(n-1)$ and $\theta(s_n) = -2\pi n$, $n \geq 1$, are well defined. Moreover, if $A_n$ are the arcs in $I_0$ with endpoints $p_n = u(t_n)$ and $q_n = u(s_n)$, then all circles $C_n = A_n \cup u([t_n, s_n])$ have winding number $-1$ around $p_N$, hence they separate $p_S$ and $p_N$. Let $R_n$ denote the open disk in $O$ enclosed by $C_n$ and construct a sequence of disks $(M_k)_{k=1}^\infty$ in $O$ such that $p_S \in M_k$ for any $k$ and $\bigcup_{k=1}^\infty M_k = O$. If $M_k$ is given, say $d(M_k, \Omega) = \delta > 0$, and $n$ is large enough such that $\max_{c \in C_n} d(c, \Omega) < \delta$, we get $C_n \cap \operatorname{Bd} M_k = \emptyset$, which together with $p_S \in M_k \cap R_n$ implies $M_k \subset R_n$. Therefore, we get $O = \bigcup_{n=1}^\infty R_n$, and recall that $\operatorname{Bd} O = \Omega$. This implies that if $q \in \Omega$ and $W$ is an arbitrarily small neighbourhood of $q$, then there is some $R_n$ (and therefore some $C_n$) intersecting $W$. Add to this that $\operatorname{diam} A_n \to 0$ to easily conclude $\Omega' = \Omega$, as we desired to prove. □

Proposition 5.24 easily follows from the previous lemmas. Namely, there are at most countably many nontrivial orbits in $O$ whose $\omega$-limit set is a singular point of $O$ (Lemma 5.30). In particular, there is $p \in O$ such that $\alpha_\Phi(p) = \{p_S\}$ and $\omega_\Phi(p)$ is not a singular point of $O$ (Lemma 5.29). Then $\omega_\Phi(p) = \Omega$ by Lemma 5.31.

# Chapter 6

# Topological classification of limit periodic sets of polynomial planar vector fields

In this chapter, our objective is to characterize topologically all limit periodic sets of polynomial families of planar vector fields. Namely, we will show that any limit periodic set is topologically equivalent to a compact connected semialgebraic set of the sphere with empty interior and, conversely, that any compact connected semialgebraic set of the sphere with empty interior can be realized as a limit periodic set.

We consider a real algebraic manifold $\Lambda$ of dimension $n \geq 1$, which we call *parameter space*. A *family of planar vector fields* $(X_\lambda)_{\lambda\in\Lambda}$, is a vector field $X_\lambda$ defined on $\mathbb{R}^2 \times \Lambda$ which is tangent to the fibres of the projection $\pi : \mathbb{R}^2 \times \Lambda \to \Lambda$. For any parameter $\lambda_0 \in \Lambda$, we denote by $X_{\lambda_0}$ the restriction of $X_\lambda$ to $\mathbb{R}^2 \times \{\lambda_0\}$, which we identify with $\mathbb{R}^2$. We say that the family $(X_\lambda)_{\lambda\in\Lambda}$ is *polynomial* if for each $\lambda_0 \in \Lambda$ there exist local coordinate systems $x = (x_1, x_2)$ of $\mathbb{R}^2$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ centered at $\lambda_0$ such that $X_\lambda(x) = A_1(x,\lambda)\partial_{x_1} + A_2(x,\lambda)\partial_{x_2}$ where $A_1$ and $A_2$ are polynomials.

Given any polynomial vector field $X$ on $\mathbb{R}^2$, we will extend it to an analytic vector field, which we denote by $\hat{X}$, in the sphere $\mathbb{S}^2$ via a Bendixson compactification (see details in Section 1.4.3). In what follows, we will be identifying $\mathbb{S}^2$ and $\mathbb{R}^2 \cup \{\infty\}$ (the one-point compactification of the plane). Also, for every $A \subset \mathbb{R}^2$, we will write $\hat{A}$ to denote the closure of $A$ seen as a subset in $\mathbb{S}^2$ of $\mathbb{R}^2$; we will say that $\hat{A}$ is the *compactification* of $A$.

The notion of limit periodic sets was first introduced by J. P. Françoise and C. Pugh [31, pp. 141]. Before presenting the definition, we recall the concept of Hausdorff distance in the context of metric spaces.

Let $(X, d)$ be a compact metric space. Given any subset $A$ and any $\epsilon > 0$, we define the $\epsilon$-neighbourhood of $A$ as $A_\epsilon = \{x \in X : d(A, x) < \epsilon\}$. Given any two nonempty closed bounded subsets of $A, B \subset X$, we then define the *Hausdorff distance between* $A$ *and* $B$ as

$$d_H(A, B) = \inf\{\epsilon > 0 : \ A \subset B_\epsilon \text{ and } B \subset A_\epsilon\}.$$

It is an easy exercise to check that $d_H$ is a metric on the set $\mathcal{C}(X)$ of all nonempty compact subsets of $X$. The topology in $\mathcal{C}(X)$ associated with $d_H$ is said to be the Hausdorff topology of $X$.

**Definition 6.1.** A *limit periodic set* for a polynomial family of planar vector fields $(X_\lambda)_{\lambda\in\Lambda}$ at the parameter $\lambda_0$ is a closed set $\Gamma \subset \mathbb{R}^2$ for which there exist a sequence $(\lambda_n)_n$ in the parameter space $\Lambda$ and a sequence $(\gamma_n)_n$ of topological circles in $\mathbb{R}^2$ such that $(\lambda_n)_n$ converges to $\lambda_0$ in $\Lambda$, $(\gamma_n)_n$ converges to $\hat{\Gamma}$ in the Hausdorff topology of $\mathbb{S}^2$ and, for every $n$, the vector field $X_{\lambda_n}$ has $\gamma_n$ as a limit cycle.

In terms of the structure of limit periodic sets, it is well-known that the Poincaré-Bendixson Theorem implies:

**Proposition 6.2.** *(See [31, Proposition 1]). Let* $(X_\lambda)_{\lambda\in\Lambda}$ *be a polynomial family of planar vector fields and* $\Gamma$ *be a limit periodic set at the parameter* $\lambda_0$*. Then* $\hat{\Gamma}$ *is one of the following:*

**(i)** *a singular point of* $\hat{X}_{\lambda_0}$*;*

**(ii)** *a periodic orbit of* $\hat{X}_{\lambda_0}$*;*

**(iii)** *a polycycle of* $\hat{X}_{\lambda_0}$ *(that is, a cyclic ordered collection of singular points* $a_1, \ldots, a_k$ *and arcs, given by integral curves, connecting them in the specific order: the* $j$*th arc connects* $a_j$ *with* $a_{j+1}$*);*

**(iv)** *a degenerate limit cycle, that is, it contains non-isolated singularities of the vector field* $\hat{X}_{\lambda_0}$*.*

While the above proposition provides some key information about the nature of limit periodic sets, it does not fully characterize them. The present chapter intends to fulfils this gap.

A first characterization was provided by D. Panazzolo and R. Roussarie in [67], under the additional hypothesis that the first jet of the singular points of $X_{\lambda_0}$ is non-vanishing. In the same paper, the authors also showed a first example of a limit periodic set which is not topologically in the list of possibilities of the Poincaré-Bendixson Theorem [67, Example 3.1]. Going further, in [10], A. Belotto presented a class of examples of limit periodic sets which, topologically, are not in the list of possibilities of Poincaré-Bendixson

Theorem either. Here, we improve and generalize the construction of [10] in order to prove the converse of the following result.

**Theorem G.** *Let $(X_\lambda)_{\lambda\in\Lambda}$ be a polynomial family of planar vector fields, $\Gamma$ be a limit periodic set for $(X_\lambda)_{\lambda\in\Lambda}$ and $\hat{\Gamma}\subset\mathbb{S}^2$ be its compactification. Then there exists a homeomorphism $\varphi:\mathbb{S}^2\to\mathbb{S}^2$ such that $\varphi(\hat{\Gamma})$ is a compact connected semialgebraic set with empty interior.*

*Conversely, if $\Gamma$ is a nonempty closed semialgebraic subset of $\mathbb{R}^2$ with empty interior whose compactification $\hat{\Gamma}\subset\mathbb{S}^2$ is connected, there exists a polynomial family of planar vector fields $(X_\lambda)_{\lambda\in\Lambda}$ having $\Gamma$ as a limit periodic set.*

*Remark* 6.3. A description of the limit periodic sets $\Gamma$ in the spirit of Proposition 6.2 follows from the proof of Lemma 6.11 below. More precisely, with the notation of the direct implication in Theorem G, a limit periodic set $\Gamma$ must be a finite union $\bigcup_{i=1}^m S_i\cup\bigcup_{j=1}^n\gamma_j$, for some $m,n\in\mathbb{N}$, where each $S_i$ is a connected semialgebraic subsets of the set of singularities of $\hat{X}_{\lambda_0}$ and each $\gamma_j$ is a regular orbit of $\hat{X}_{\lambda_0}$ which converge to a singular points in $\bigcup_{i=1}^m S_i$. Even more, each $\gamma_j$ is characteristic in both extremes; that is, when the orbit is run in either negative or positive time, the orbit converges in a definite direction to a singular point of $\hat{X}_{\lambda_0}$ and, in a sufficiently small neighbourhood of that limit point, $\gamma_j$ is the boundary of a parabolic or hyperbolic sector.

*Remark* 6.4. If we restrict our study to compact limit periodic sets of the plane, Theorem G can be extended to the analytic category. More precisely, with the same ideas and techniques, it is not difficult to show that a compact limit periodic set for an analytic family of vector fields is topologically equivalent to a compact connected semianalytic set with empty-interior; conversely, every compact connected semianalytic set with empty-interior can be realized as a limit periodic set for an analytic family of vector fields.

On the other hand, Theorem G does not extend, in a trivial, to unbounded limit periodic sets for families of analytic vector fields. The difficulty relies on proving the converse part of the theorem. Let us first exemplify how our methods could be adapted to some unbounded analytic varieties: we claim that the set

$$\Gamma_1=\{(x,y)\in\mathbb{R}^2;\ f_1(x,y)=y^2-\sin(x)^2=0\}$$

can be realized as a limit periodic set for an analytic family. Indeed, it suffices to replace the function $h$ in Section 6.2.2 by

$$h(x,y,\lambda,\alpha)=f_1(x,y)^2-\lambda\left(1-\alpha^2(x^2+y^2)\right).$$

We leave it to the reader to verify that the ideas of Sections 6.2.2 and 6.2.3 can be adapted to this function. Nevertheless, it is unclear which connected subsets of

$$\Gamma_2 = \{(x, y) \in \mathbb{R}^2;\ f_2(x, y) = y^3 - y\sin(x)^2 = 0\}$$

can be realized as limit periodic sets. Technically, the difficulty is that our construction for $\Gamma_2$ would demand the use of transition points (defined in the last paragraph of Section 6.2.1); but the set of those transition points $\mathrm{Tr}(\Gamma_2)$ would need to be infinite in this case.

The following example illustrates the construction performed in Section 6.2 to prove the converse of Theorem G.

**Example 6.5.** Let $\Gamma \subset \mathbb{R}^2$ be the semi-algebraic set given by:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2;\ f(x, y) = y(x^2 + y^2 - 1) = 0 \text{ and } g(x, y) = x^2 + y^2 \leq 4\right\}.$$

Following the notation of Subsection 6.2.2, we consider the set of points

$$S = \{(-2, 0), (2, 0), (0, 1), (0, -1)\}$$

(where notice that $S = \mathrm{Gen}(\Gamma) \cup \mathrm{Tr}(\Gamma)$ and $NG(\Gamma) = \emptyset$, see Definition 6.12). Now, consider the three variable polynomial

$$h(x, y, \lambda) = f(x, y)^2 - \lambda \prod_{p \in S} \left(\|(x, y) - p\|^2 - \lambda^2\right)$$

where $\lambda$ will play the role of the parameter of the family of vector fields. Let $t \in \mathbb{R}^+$ and note that the level curves $Z_t = \{(x, y) \in \mathbb{R}^2;\ h(x, y, t) = 0\}$ are connected and converge (in the Hausdorff topology) to $\Gamma$ when $t$ goes to zero (c.f. Proposition 6.19; see Figure 6.1). It follows that the perturbation of the Hamiltonian vector field given by

$$X_\lambda = \left(\frac{\partial h}{\partial y} + h\frac{\partial h}{\partial x}\right)\partial_x + \left(-\frac{\partial h}{\partial x} + h\frac{\partial h}{\partial y}\right)\partial_y$$

is an polynomial family of planar vector fields which has $\Gamma$ as a limit periodic set for the parameter $\lambda_0 = 0$ (for every $t > 0$, the set $Z_t$ is a limit cycle for $X_t$).

The rest of the chapter is divided as follows: the aim of Section 6.1 is to prove the direct implication of Theorem G while Section 6.2 deals with the converse one.

## 6.1 Topology of limit periodic sets

Subsection 6.1.1 is a preliminary section, devoted to recall the notion of real semialgebraic set and some of its elementary properties, while in Subsection 6.1.2 we present the proof of the direct part of Theorem G.

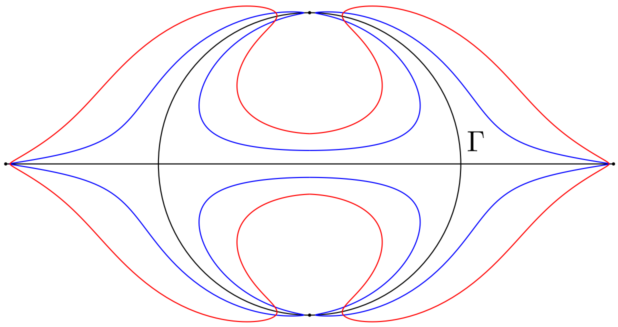


Figure 6.1: Limit cycles for $t = 0.001$ (red) and $t = 0.0001$ (blue) approaching the limit periodic set $\Gamma$.

### 6.1.1 Semialgebraic sets

Let $x = (x_1, \ldots, x_n)$ be a coordinate system of $\mathbb{R}^n$. Given any polynomial $f$ on $\mathbb{R}^n$ we will say that $(f(x) = 0) = \{x \in \mathbb{R}^n : f(x) = 0\}$ is an *algebraic set*. A more general concept is the following one.

**Definition 6.6.** (See [14, Section 1]). A subset $Z \subset \mathbb{R}^n$ is *semialgebraic* if there exist polynomials $f_i$ and $g_{ij}$ on $\mathbb{R}^n$, $i = 1, \ldots, p$ and $j = 1, \ldots, q$, such that

$$Z = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in \mathbb{R}^n : f_i(x) = 0 \text{ and } g_{ij}(x) > 0\}.$$

A set $Z \subset \mathbb{S}^2 \subset \mathbb{R}^3$ is said to be semialgebraic if $Z$ is a semialgebraic set of $\mathbb{R}^3$.

A fist collection of examples of semialgebraic sets is given by the finite unions of arcs linear by parts. For every $a, b \in \mathbb{R}^2$, we will denote $[a, b] = \{a + sb : 0 \leq s \leq 1\}$, the straight arc joining $a$ and $b$. Let $a_1, \ldots, a_n$ be points in $\mathbb{R}^2$ and call $l_j = [a_j, a_{j+1}]$ for any $1 \leq j \leq n-1$. If $l_j \cap l_{j'} = \emptyset$ when $|j - j'| \neq 1$ and $l_j \cap l_{j+1} = \{a_{j+1}\}$ for any $1 \leq j \leq n-1$, we say that $L = \cup_{j=1}^{n-1} l_j$ is an *arc linear by parts*. The points $a_1$ and $a_n$ are said to be the *endpoints* of $L$.

Every connected generalized graphs in $\mathbb{S}^2$ (recall the definition in page 37) is topologically equivalent to a semialgebraic set. This fact will play an important role in the proof of Theorem G.

**Lemma 6.7.** *Let $L \subset \mathbb{S}^2$ be a connected generalized graph. Then there exists a homeomorphism of $\mathbb{S}^2$ onto itself taking $L$ to a semialgebraic set.*

*Proof.* Let us start by noticing that, apart from composing a rotation with the transition homeomorphism $\phi : \mathbb{R}^2_\infty \to \mathbb{R}^2_\infty$ associated with the Bendixson compactification (see Section 1.4.3), we may suppose that there exists a compact connected set $\Gamma \subset \mathbb{R}^2$ such that its completion in $\mathbb{R}^2_\infty$ is equal to $L$. Also, if we call $T \subset \Gamma$ the subset of points which are not star points of order 2, then $T$ is a finite set. It is then enough to prove the result under the hypothesis of $\Gamma$ being nonempty.

If $T$ is empty, there is nothing to say: $L$ is a circle [54, Theorem 2, p. 180]. Otherwise, let us say that $T = \{a_1, \ldots, a_m\}$ for some $m \geq 1$. For every $1 \leq j \leq m$ we can take a neighbourhood $B_j \subset \mathbb{R}^2$ of $a_j$ such that $B_j \cap T = \{a_j\}$ and $B_j \cap \Gamma$ is an $n_j$-star. Without lost of generality we can also assume that, for every $1 \leq j \leq m$, $B_j$ is a standard euclidean compact ball of center $a_j$ in $\mathbb{R}^2$, that $\partial B_j$ meets $\Gamma$ in exactly $n_j$ points $b_{j,1}, \ldots, b_{j,n_j}$ and, as a consequence, $B_j \cap \Gamma$ is homeomorphic to $M_j = \cup_{k=1}^{n_j}[a_j, b_{j,k}]$. Now any of the components of $\Gamma \setminus \cup_{j=1}^{m} B_j$ is a generalized graph consisting only of star points of order 2, let us say $U_1, \ldots, U_\tau$ are those components. For any $1 \leq k \leq \tau$, we can take an arc linear by parts $N_k$ whose endpoints coincide with the two points in $\mathrm{Cl}(U_k) \setminus U_k$ and such that $\cup_{j=1}^{m} M_j \cup \cup_{k=1}^{\tau} N_k$ is homeomorphic to $\Gamma$. This last homeomorphism can be extended to a homeomorphism from the sphere to the sphere (see Section 1.1.4). $\square$

### 6.1.2 Topological properties of periodic limit sets

Let $(X_\lambda)_{\lambda \in \Lambda}$ be a polynomial family of planar vector fields and, for every $\lambda \in \Lambda$, let $\hat{X}_\lambda$ be the analytic vector field on $\mathbb{S}^2$ described by the Bendixson compactification as in Section 1.4.3 (we remark that $p_N = (0, 0, 1)$ is a singular point for every $\hat{X}_\lambda$). Together with the family $(\hat{X}_\lambda)_{\lambda \in \Lambda}$ we may consider the associated analytic flow $\Phi : \mathbb{R} \times \mathbb{S}^2 \times \Lambda \to \mathbb{S}^2$.

The continuity of the flow already gives some topological and dynamical obstructions for the limit periodic sets: a limit periodic set at a parameter $\lambda_0$ must be invariant for $X_{\lambda_0}$ and its compactification by one point must be connected.

**Lemma 6.8.** *If $\Gamma$ is a limit periodic set at the parameter $\lambda_0$, then $\hat{\Gamma}$ is connected and invariant for $\hat{X}_{\lambda_0}$ (equivalently, $\Gamma$ is invariant for $X_{\lambda_0}$).*

*Proof.* Let us start fixing a sequence in $\Lambda$ converging to $\lambda_0$, $(\lambda_n)_n$, and a sequence of topological circles in $\mathbb{R}^2$, $(\gamma_n)_n$, such that $(\hat{\gamma}_n)_n$ converges to $\hat{\Gamma}$ in the Hausdorff topology of $\mathbb{S}^2$ and, for every $n$, $\gamma_n$ is a limit cycle of $X_{\lambda_n}$.

Firstly, to prove the invariance of $\hat{\Gamma}$, we consider points $a \in \Gamma$ and $b = \Phi(s, a, \lambda_0)$ for some $s \in \mathbb{R}$ and a sequence of points $a_n \in \gamma_n$ converging to $a$. By the continuity of $X_\lambda$, the points $\Phi(s, a_n, \lambda_n)$ converge to $b$ so $b \in \hat{\Gamma}$.

Next, to obtain a contradiction, let us suppose that $\hat{\Gamma}$ is not connected and choose two disjoint open sets $V_1$ and $V_2$ of $\mathbb{S}^2$ which disconnect $\hat{\Gamma}$. Since $\gamma_n \to \hat{\Gamma}$ in the Hausdorff topology, we conclude that $\gamma_n \subset V_1 \cup V_2$, $\gamma_n \cap V_1 \neq \emptyset$ and $\gamma_n \cap V_2 \neq \emptyset$, for every sufficiently large $n$. But this implies that $\gamma_n$ is disconnected, which is impossible. $\square$

From the analyticity of the flow $\Phi$ (we only need to use that it is of class $C^1$), the following important local property is established: any limit periodic set can meet at most once with any traversal. We formalize this property below.

If $a \in \mathbb{S}^2$ is a regular point of $\hat{X}_{\lambda_0}$, then we can always find a positive real number $\varepsilon > 0$ and an analytic embedding $\sigma : (-\varepsilon, \varepsilon) \to \mathbb{S}^2$ being a transversal section of $\hat{X}_{\lambda_0}$ with $\sigma(0) = a$. On the other hand, given any transversal section of $\hat{X}_{\lambda_0}$, $\sigma : I \to \mathbb{S}^2$, it is clear that for any $t \in I$ we can take $I(t)$, a neighbourhood of $t$ in $I$, and $\Lambda(\lambda_0)$, a neighbourhood of $\lambda_0$ in $\Lambda$, such that the restriction of $\sigma$ to $I(t)$ is a transversal section of $\hat{X}_\lambda$ for every $\lambda \in \Lambda(\lambda_0)$. These observations, together with the *Flow Box Theorem* (see Theorem 1.51) and the fact that any periodic orbit of a $C^1$ vector fields on the sphere can meet any transversal section only once (see, e.g., [66, p. 16]), give the following result.

**Lemma 6.9.** *(See [72, Lemma 2, p. 20]). Let $\Gamma$ be a limit periodic set at the parameter $\lambda_0$. Then any transversal section of $\hat{X}_{\lambda_0}$ meets $\hat{\Gamma}$ at most once.*

The following auxiliary result is a distinguished property of analytic flows on open subsets of the sphere. Its proof, which can be found, for example, in [46, Lemma 3.2], needs the combination of both Theorem A and Remark 2.1.

**Lemma 6.10.** *Let $\Phi$ be an analytic local flow on a region $O \subset \mathbb{S}^2$ and $A \subset O$ be an analytic set. For every $p \in O$, if $\omega_\Phi(p) = \{q\}$ for an isolated singular point $q$, then either $q$ is an attracting focus, $\varphi(p) \subset A$ or $\varphi(p, +) \cap A$ is discrete.*

The last ingredient we need is the well-known behaviour of analytic vector fields on the neighbourhood of isolated singular points. An isolated singular point of an analytic vector field is either an attracting or repulsing focus, a center of it has the finite sectorial decomposition (see p. 34).

We are now ready to prove the direct implication of Theorem G. The work is done by the combination of Lemma 6.7 and the following result.

**Lemma 6.11.** *If $\Gamma$ is a limit periodic set at the parameter $\lambda_0$, then $\hat{\Gamma}$ is a connected generalized graph.*

*Proof.* According with Lemma 6.8, $\hat{\Gamma}$ is a connected subset of $\mathbb{S}^2$ which is a union of orbits of $\hat{X}_{\lambda_0}$. Therefore, we only need to prove that all the points of $\hat{\Gamma}$ are star points. We fix a point $a \in \hat{\Gamma}$ and distinguish three cases.

If $a$ is a regular point of $\hat{X}_{\lambda_0}$, the *Flow Box Theorem* and Lemma 6.9 imply the existence of a neighbourhood of $a$, $U_a$, such that $\hat{\Gamma} \cap U_a$ is a 2-star.

Let us now assume that $a$ is an isolated singular point of $\hat{X}_{\lambda_0}$ and let $U_a$ be a neighbourhood of $a$ such that every point in $U_a \setminus \{a\}$ is a regular point of $\hat{X}_{\lambda_0}$. If $a$ is a center (respectively a focus) for $\hat{X}_{\lambda_0}$, we can always find a transversal section accumulating at $a$ and meeting at least once (respectively twice) any regular orbits of $\hat{X}_{\lambda_0}$ in $U_a$ so Lemma 6.9

guarantees that, after shrinking $U_a$ if necessary, $\hat{\Gamma} \cap U_a = \{a\}$. Otherwise, we can consider characteristics orbits $c_0, \ldots, c_{n-1}$, with $n \geq 2$, defining a sectorial decomposition around $a$ (we follow the notation of Section 1.4.3). Using once again Lemma 6.9, we note that: at each parabolic sector there may exist only one regular orbit contained in $\hat{\Gamma}$; at each hyperbolic sector, apart from shrinking $U_a$, the intersection with $\hat{\Gamma}$ can only be the characteristic orbits $c_j$ defining this sector; at each elliptic sector, apart from shrinking $U_a$ and adding two new parabolic sectors, we may suppose that the intersection of the elliptic sector with $\hat{\Gamma}$ is empty. It follows from these observations that, also in this case, $\hat{\Gamma} \cap U_a$ is a star of vertex $a$.

Finally, if $a$ is a non-isolated singularity of $\hat{X}_{\lambda_0}$, it is well known that there exist a neighbourhood of $a$, $U_a$, an analytic map $f : U_a \to \mathbb{R}$ and an analytic vector field $Y$ on $U_a$ such that the restriction of $\hat{X}_{\lambda_0}$ to $U_a$ coincides with the product $f\,Y$ and the vector field $Y$ has no zeros in $U_a \setminus \{a\}$ (see Proposition 1.9).

Let us denote by $Z$ the analytic set $f^{-1}(0)$ and note that, after shrinking $U_a$ if necessary, $Z$ is a star (with $a$ as vertex) decomposing $U_a$ into finitely many connected components any of which contains no singular points of $\hat{X}_{\lambda_0}$. Furthermore, by analyticity, there is no loss of generality in assuming that the neighbourhood $U_a$ has been chosen such that each branch of $Z$ is either invariant by $Y$ or a transversal section of $Y$ (see Lemma 6.10). Accordingly, $\hat{\Gamma} \cap U_a$ is the union of $\{a\}$ with some of the branches of $Z$ and some regular orbits of $Y$.

The above observation allow us to adapt the argument given in the first two cases, mutatis mutandis, to the case when $a$ is a regular point of $Y$, or when $Y$ admits a sectorial decomposition at $a$ (where we are again considering at least two characteristic orbits and among them appear at least all the branches of $Z \setminus \{a\}$ which are invariant by $Y$). We remark that the latter case includes the scenario of $a$ being a node point for $Y$. Finally, if $a$ is a center or a focus point of $Y$, it is elementary to show that in any of the connected components of $U_a \setminus Z$ there exists a transversal section accumulating at $a$ and at the boundary of $U_a$. Consequently, in these two cases, it may be conclude that, after shrinking $U_a$ if necessary, $\hat{\Gamma} \cap U_a \subset Z$ and $a$ is a star point. $\square$

Clearly, the direct implication of Theorem G follows from Lemmas 6.7 and 6.11.

## 6.2 Construction of limit periodic sets: proof of converse part of Theorem G

### 6.2.1 Properties of semialgebraic sets

We are interested in planar semialgebraic sets with empty interior. Associated with any of these sets, we may introduce a free-square polynomial whose set of zeros will play an important role in the rest of the chapter.

Let us start fixing a coordinate system for the plane $x = (x_1, x_2)$ and let $\Gamma \subset \mathbb{R}^2$ be a semialgebraic set with empty interior and whose compactification $\hat{\Gamma}$ is connected. If $\Gamma$ is itself an algebraic set we simply take a free-squared polynomial $f_\Gamma$ making $(f_\Gamma(x) = 0) = \Gamma$. Assume now that $\Gamma$ is not an algebraic set; in particular, and because $\hat{\Gamma}$ is connected, we note that none of the components of $\Gamma$ can be singletons. Let $f_i$ and $g_{i,j}$, $1 \leq i \leq p$ and $1 \leq j \leq q$, be polynomials such that

$$\Gamma = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in \mathbb{R}^2 : \, f_i(x) = 0 \text{ and } g_{ij}(x) > 0\}. \tag{6.1}$$

Without lost of generality, we can assume that all the $f_i$ are irreducible and also, because $\Gamma$ has empty interior, that all of them are non-constant and $(f_i(x) = 0) \cap \Gamma$ is one dimensional. Using the well-known fact that any two co-prime polynomials on $\mathbb{R}^2$ can meet only finitely many times (see, for example, [20, Lemma 4]), it is not difficult to reason that under such conditions the polynomials $f_i$ in (6.1) are uniquely defined (up to the multiplication of non-zero constants). Let us take $f_\Gamma$ as the free-square polynomial associated with the product $\prod_{i=1}^{p} f_i$; this polynomial verifies $\Gamma \subset (f_\Gamma(x) = 0)$ and is uniquely defined from (6.1) in the terms just expressed. In any of the two cases discussed above, we will refer to the polynomial $f_\Gamma$ as *the polynomial associated with* $\Gamma$. The set of zeros of $f_\Gamma$, which we will denote by $A_\Gamma = (f_\Gamma(x) = 0)$, will be also said to be *the algebraic set associated with* $\Gamma$.

**Definition 6.12.** Let $\Gamma \subset \mathbb{R}^2$ be a semialgebraic set with empty interior and such that $\hat{\Gamma}$ is connected and let $f_\Gamma$ and $A_\Gamma$ be its associated polynomial and algebraic set respectively. A point $a \in \Gamma$ is said to be:

(1) an *algebraic* point of $\Gamma$ if there exists a neighbourhood of $a$ in $\mathbb{R}^2$, $U$, such that $U \cap \Gamma = U \cap A_\Gamma$. We denote the set of algebraic points of $\Gamma$ by $Alg(\Gamma)$;

(2) a *generic non-algebraic* point of $\Gamma$ if $a \notin Alg(\Gamma)$ and $A_\Gamma$ is regular at $a$ (i.e. the gradient of $f_\Gamma$ at $a$ is non-zero). We denote the set of generic non-algebraic points of $\Gamma$ by $\mathrm{Gen}(\Gamma)$;

(3) a *non-generic non-algebraic* point of $\Gamma$ if $a \notin Alg(\Gamma)$ and $A_\Gamma$ is singular at $a$ (i.e. the gradient of $f_\Gamma$ vanishes at $a$). We denote the set of non-generic non-algebraic point of $\Gamma$ by $NG(\Gamma)$.

*Remark* 6.13. The sets of non-algebraic points $\mathrm{Gen}(\Gamma)$ and $NG(\Gamma)$ are both finite.

*Remark* 6.14. Let us assume that $NG(\Gamma)$ is nonempty, say $NG(\Gamma) = \{a_1, \ldots, a_r\}$ for some positive integer $r$. For every $k \in \{1, \ldots, r\}$, take a sufficiently small euclidean ball $B_k = B(a_k, \rho_k)$ centered at $a_k$ with radius $\rho_k > 0$ and denote by $n_k$ the number of connected components of $(A_\Gamma \setminus \Gamma) \cap B_k$ (the number $n_k$ is the same for every sufficiently small $\rho_k > 0$). By *Newton-Puisseux Theorem* (c.f. Section 2), for every $k \in \{1, \ldots, r\}$, there exist sequences of points $(a_i^{j,k})_{i\in\mathbb{N}} \subset A_\Gamma \setminus \Gamma$, $j \in \{1, \ldots, n_k\}$, such that each sequence is contained in a different connected component of $(A_\Gamma \setminus \Gamma) \cap B_k$ and $a_i^{j,k} \to a_j$ when $i \to \infty$.

The following objects are used in Section 6.2.3: the number $n_\Gamma = \sum_{k=1}^{r} n_k$; the sequence of points in $\mathbb{R}^{2n_\Gamma}$, $(\alpha_i)_{i\in\mathbb{N}}$, given by $\alpha_i = (a_i^{1,1}, \ldots, a_i^{n_1,1}, \ldots, a_i^{1,r}, \ldots, a_i^{n_r,r})$; and the limit of $(\alpha_i)_i$, $\alpha_0 \in \mathbb{R}^{2n_\Gamma}$.

*Remark* 6.15. It follows from Remark 6.14 that there exists a sequence of semialgebraic sets with empty interior $(\Gamma_i)_{i\in\mathbb{N}}$ such that $\Gamma \subset \Gamma_i \subset A_\Gamma$, $NG(\Gamma_i) = \emptyset$ and $\Gamma_i \to \Gamma$ (in the Hausdorff topology) when $i \to \infty$. Moreover, the polynomials $f_\Gamma$ and the algebraic set $A_\Gamma$ are also the polynomial and the algebraic set associated with any of those $\Gamma_i$ and $\mathrm{Gen}(\Gamma_i) = \mathrm{Gen}(\Gamma) \cup \{a_i^{1,1}, \ldots, a_i^{n_1,1}, \ldots, a_i^{1,r}, \ldots, a_i^{n_r,r}\}$.

Now assume that $\Gamma$ is compact and connected. There exists a finite number of (non-unique) points $b_1, \ldots, b_k \in Alg(\Gamma)$ which are regular points of the algebraic set $A_\Gamma$ and such that both $\Gamma \setminus \{b_1, \ldots, b_k\}$ and $\mathbb{R}^2 \setminus (\Gamma \setminus \{b_1, \ldots, b_k\})$ are connected. We can always fix a certain number of these points, which we call *transition points*, and denote their set by $\mathrm{Tr}(\Gamma)$. We remark that the minimal number $k$ of transition points corresponds to the number of connected components of $\mathbb{R}^2 \setminus \Gamma$ minus one. Moreover, with the notation of Remark 6.15, the set $\mathrm{Tr}(\Gamma)$ is a valid set of transition points for $\Gamma_i$, for all $i$ sufficiently big.

### 6.2.2 Construction of generic compact limit periodic sets

Let us fix a compact connected semialgebraic set $\Gamma \subset \mathbb{R}^2$ with empty interior and such that $NG(\Gamma) = \emptyset$. Let $f = f_\Gamma$ be the polynomial associated with $\Gamma$ and $A_\Gamma$ its set of zeros and fix a transition set for $\Gamma$, $\mathrm{Tr}(\Gamma)$. Fix a coordinate system $x = (x_1, x_2)$ of $\mathbb{R}^2$ and a parameter $\lambda \in \mathbb{R}$. Denote by $S$ the finite set $\mathrm{Gen}(\Gamma) \cup \mathrm{Tr}(\Gamma)$.

We consider the function

$$h(x, \lambda) = f(x)^2 - \lambda \prod_{p\in S} \left(\|x - p\|^2 - \lambda^2\right),$$

where $\|\cdot\|$ stands for the euclidean norm on $\mathbb{R}^2$, and the polynomial family of planar vector fields $(X_\lambda)_{\lambda\in\mathbb{R}}$ given by

$$X_\lambda = \left(\frac{\partial h}{\partial x_2} + h\frac{\partial h}{\partial x_1}\right)\partial_{x_1} + \left(-\frac{\partial h}{\partial x_1} + h\frac{\partial h}{\partial x_2}\right)\partial_{x_2}. \tag{6.2}$$

We devote the rest of the section to show $\Gamma$ is a limit periodic set of $(X_\lambda)_{\lambda\in\mathbb{R}}$ at $\lambda = 0$. The key to achieve this is to understand how the level curves (in respect to the parameter

$\lambda$) of $h$ are. We will start giving a local description of $(h(x,\lambda)=0)$ in a neighbourhood of a point $(a,0)$ with $a\in\Gamma$; we treat separately the cases $a\in Alg(\Gamma)\setminus \mathrm{Tr}(\Gamma)$ (Lemma 6.17) and $a\in \mathrm{Gen}(\Gamma)\cup \mathrm{Tr}(\Gamma)$ (Lemma 6.18).

Let us begin with an auxiliary result (an application of the the local structure of analytic sets).

**Lemma 6.16.** *Let $U$ be an open and connected subset of the plane and $f: U\subset\mathbb{R}^2\to\mathbb{R}$ be a non-constant analytic map. Then the set of critical values of $f$ is discrete (by critical value we understand a point $b\in\mathbb{R}$ such that the level set $f^{-1}(b)$ contains points where the gradient of $f$ vanishes.).*

*Proof.* Let us proceed by contradiction assuming the existence of a point $z\in U$ and a sequence $(z_n)_n$ of zeros of $\nabla f$ converging to $z$ and such that $\{f(z_n) : n\in\mathbb{N}\}$ is infinite. In other words, if we consider the analytic function $g: U\to\mathbb{R}$ given by

$$g(z)=\left(\frac{\partial f(z)}{\partial x}\right)^2+\left(\frac{\partial f(z)}{\partial y}\right)^2,$$

we are assuming the existence of a sequence of zeros of $g$ accumulating in its domain and whose images under $f$ produces an infinite set. Since $g^{-1}(0)$ is a generalized graph, we deduce that the set of zeros of $g$ contains a whole open arc $L$ where $f$ is not constant. The continuity of the composition of any continuous parametrization of $L$ as a curve and $f$ allows us to conclude that the set of critical values of $f$ contains at least an arc but this is in contradiction with *Sard's theorem* (see, e.g., [55, Theorem 6.10]). □

Here and subsequently, given any set $A\subset\mathbb{R}^2\times\mathbb{R}$ and any $t\in\mathbb{R}$ we will denote $A\cap(\lambda=t)=\{(x,\lambda)\in A:\lambda=t\}$; when convenient, we will also understand that $A\cap(\lambda=t)$ is identified with $\{x\in\mathbb{R}^2:(x,t)\in A\}$. In particular, the set $Z_t$, which stands for the level curve $(h(x,\lambda)=0)\cap(\lambda=t)$, will repeatedly be seen as a subset of $\mathbb{R}^2$.

**Lemma 6.17.** *For every $a\in Alg(\Gamma)\setminus \mathrm{Tr}(\Gamma)$ there exist a number $\epsilon_a>0$ and a compact neighbourhood $V_a$ of $a$ such that $Z_t\cap V_a\subset\mathbb{R}^2\setminus\Gamma$ for every $0<t<\epsilon_a$. Moreover, for any connected component $W$ of $V_a\setminus\Gamma$ and any $0<t<\epsilon_a$, $Z_t\cap W$ is a nonempty connected regular curve which converges (in the Hausdorff topology) to $\mathrm{Cl}(W)\cap\Gamma$, when $t$ tends to $0$ (see Figure 6.2).*

*Proof.* Let us start considering a number $\epsilon_a>0$, a compact neighbourhood $V_a\subset\mathbb{R}^2$ of $a$ and the coordinate system $z=x-a$ (which is centered at $a$) such that $h(z,\lambda)=f(z)^2-\lambda u(z,\lambda)$, where $u(z,\lambda)>0$ at all points in $V_a\times(\epsilon_a,-\epsilon_a)$.

By the *implicit function theorem*, we may assume that there exists an analytic function $\lambda: V_a\to\mathbb{R}$ such that $h(z,\lambda(z))=0$ for every $z\in V_a$.

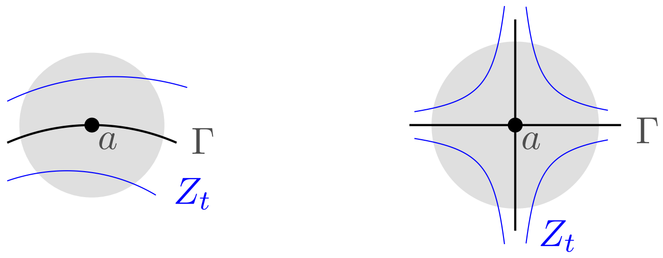


Figure 6.2: A regular point (left) and a singular point (right).

We note that the curves $Z_t^a = Z_t \cap V_a$ correspond to the $t$-level curves of $\lambda(z)$, that is, $Z_t^a = (\lambda(z) = t)$. By continuity of $\lambda(z)$, shrinking $V_a$ if necessary, this implies that $Z_t^a$ converges (in the Hausdorff topology) to $\Gamma \cap V_a = (\lambda(z) = 0)$. Furthermore, since the level curves of non-constant analytic functions (restricted to a compact set) are generically regular (see Lemma 6.16 above), we conclude that $Z_t^a$ are regular for all $t > 0$ sufficiently small.

Next, since $\lambda(z) \geq 0$ for every $z \in V_a$, we conclude that, for every component $W$ of $V_a \setminus \Gamma$, $Z_t^a \cap W$ is nonempty for all small enough $t > 0$.

Finally, fix a component $W$ of $V_a \setminus \Gamma$ and suppose by contradiction that there exists a sequence $(t_n)_n$ converging to 0 and such that $Z_{t_n}^a \cap W$ is not connected. Without loss of generality, we may suppose that $\mathrm{Cl}(W)$ is a compact semialgebraic set. Denote by $\Gamma_W$ the semialgebraic set $\mathrm{Cl}(W) \cap \Gamma$. By the curve selection Lemma (see for example [61, Lemma 3.1]), there exists an analytic curve $\phi : [0,1] \to \mathrm{Cl}(W)$ such that $\phi(1) = a \in \Gamma_W$, $\phi(t) \in \mathrm{Cl}(W) \setminus \Gamma_W$ for all $t \neq 0$ and $\mathrm{Cl}(W) \setminus \phi([0,1])$ is not connected. Since all connected components of $Z_t^a$ must converge to $\Gamma_W$, we conclude that the curve $\phi([0,1])$ intersects each of the components of $Z_t^a$. This implies that the function $\lambda \circ \phi$ is constant and equal to 0 (the value at $\phi(1)$), which is a contradiction. $\square$

**Lemma 6.18.** *For every $a \in \mathrm{Gen}(\Gamma) \cup \mathrm{Tr}(\Gamma)$, there exist a neighbourhood $V_a$ of $a$, a positive $\epsilon_a > 0$ and a coordinate system $(y, \lambda)$ defined on $V_a \times (-\epsilon_a, \epsilon_a)$ and centered at $(a, 0)$ such that $A_\Gamma \cap V_a = (y_1 = 0)$ and*

$$h(y, \lambda) = u(y, \lambda)\left[y_1^2 - \lambda\left(y_2^2 - \lambda^2\right)\right] \tag{6.3}$$

*where $u(y, \lambda)$ is a unit over $V_a \times (-\epsilon_a, \epsilon_a)$ (see Figure 6.3).*

*Proof.* Consider the coordinate system $z = x - a$ (which is centered at $a$) and note that in a sufficiently small neighbourhood of $(a, 0)$ of the form $U_a = V_a \times (-\epsilon_a, \epsilon_a)$, we can write

$$h(z, \lambda) = f(z)^2 - \lambda\left[z_1^2 + z_2^2 - \lambda^2\right] u(z, \lambda)$$

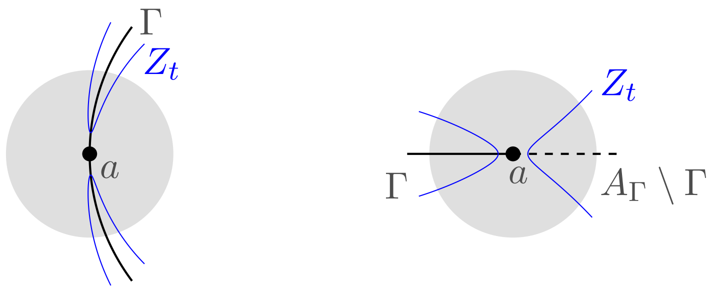


Figure 6.3: A transition point (left) and a generic point (right).

where $u(z,\lambda) > 0$ at all points in $U_a$. Apart from shrinking $U_a$, we can suppose that $\nabla f(z) \neq 0$ at all points in $U_a$. Therefore (apart from a preliminary rotation) the change of coordinates $\widetilde{y}_1 = u(z,\lambda)^{-\frac{1}{2}} f(z) = \xi z_1 + \psi(z,\lambda)$ (where $\xi \neq 0$ and $\psi(z,\lambda)$ has order at least two) and $\widetilde{y}_2 = z_2$ is an isomorphism on $U_a$. We get

$$h(\widetilde{y},\lambda) = u(\widetilde{y},\lambda)\left(\widetilde{y}_1^2 - \lambda\left(\widetilde{y}_1^2 + \widetilde{y}_2^2 v\left(\widetilde{y},\lambda\right) - \lambda^2\right)\right)$$

where $v(\widetilde{y},\lambda)$ is an analytic function such that $v(0,0) > 0$. Finally, apart from shrinking $U_a$, the change of coordinates $y_1 = \widetilde{y}_1\sqrt{1+\lambda}$ and $y_2 = \widetilde{y}_2\sqrt{v(\widetilde{y},\lambda)}$ is an isomorphism making

$$h(y,\lambda) = u(y,\lambda)\left[y_1^2 - \lambda\left(y_2^2 - \lambda^2\right)\right]$$

and $(y_1 = 0) = (\widetilde{y}_1 = 0) = (f(z) = 0) \cap V_a$ as we wanted to prove. □

**Proposition 6.19.** *There exist an open neighbourhood $U$ of $\Gamma \times \{0\}$ and a number $\epsilon > 0$ such that, for every $0 < t < \epsilon$, $Z_t \cap U$ contains a compact and regular connected component $\gamma_t$ such that $\gamma_t \to \Gamma$ (in the Hausdorff topology) when $t \to 0$.*

*Proof.* For every $a \in \Gamma$ take a neighbourhood $V_a$ of $a$ and a number $\epsilon_a > 0$ as in Lemma 6.17 or 6.18. The compacity of $\Gamma$ allows us to take a relatively compact open neighbourhood $U$ of $\Gamma \times \{0\}$, of the form $U = V \times (-\delta, \delta)$ with $V \subset \mathbb{R}^2$ and $\delta > 0$, such that $U \subset \bigcup_{a \in \Gamma} U_a$.

Note that, from the two previous lemmas, we can assume that $Z_t \cap U$ is regular for every sufficiently small $t > 0$. Also, the continuity of $h$ guarantees that $Z_t \cap U$ converges to $A_\Gamma \cap V$ when $t$ tends to $0$ (in the Hausdorff topology).

Let us fix a point $b \in Alg(\Gamma) \setminus \mathrm{Tr}(\Gamma)$ and $W$ a component of $V_b \setminus \Gamma$. For every sufficiently small $t > 0$, let us call $\gamma_t$ the connected component of $Z_t$ which meets $W$ (see Lemma 6.17). Let $\gamma_0 \subset A_\Gamma$ denote the limit of $\gamma_t$ when $t \to 0$ (which contains $b \in \Gamma$). We are then left with the task of proving that $\gamma_0 = \Gamma$.

We start showing that $\gamma_0 \subset \Gamma$. We proceed by contradiction assuming the existence of a point $c \in \gamma_0 \setminus \Gamma$. After shrinking $V$ and $V_a$ for $a \in \mathrm{Gen}(\Gamma)$ if necessary, we may suppose that the points $b$ and $c$ lies in different connected components of the set $R = V \setminus \bigcup_{a\in \mathrm{Gen}(\Gamma)} V_a$. In particular, $\gamma_t \cap R$ is disconnected and these disconnected components can only join each other by passing through one of the open sets $U_a$ with $a \in \mathrm{Gen}(\Gamma)$. This leads to contradiction with Lemma 6.18.

Since $\gamma_t$ is a connected regular curve, $\mathbb{R}^2 \setminus \gamma_t$ must consists in exactly two connected components, say $C_t^1$ and $C_t^2$. Now, for every $\epsilon_0 > 0$, consider the set $\gamma_{\epsilon_0} = \gamma_0 \setminus \cup_{a\in \mathrm{Tr}(\Gamma)} B(a, \epsilon_0)$. We claim that, for every small enough $t > 0$,

$$\gamma_{\epsilon_0} \subset \mathbb{R}^2 \setminus \gamma_t. \tag{6.4}$$

Indeed, let us suppose that $\gamma_{\epsilon_0}$ meets both $C_t^1$ and $C_t^2$ for all small $t > 0$. Since $\gamma_t$ can only cross points of $\Gamma$ near $a \in \mathrm{Tr}(\Gamma) \cup \mathrm{Gen}(\Gamma)$, we conclude that there exists a point $a \in \mathrm{Tr}(\Gamma)$ such that $\gamma_t$ crosses $\Gamma \cap V_a$ in such a way that $\gamma_{\epsilon_0} \cap C_t^1 \cap V_a$ and $\gamma_{\epsilon_0} \cap C_t^2 \cap V_a$ $\Gamma$ are both nonempty. But this gives us again a contradiction with Lemma 6.18.

Finally, let us denote by $W_i$, $i = 1, \dots k$, the connected components of $\mathbb{R}^2 \setminus \Gamma$ and set $I$ as the subset of indexes $i \in \{1, \dots, k\}$ such that $W_i \cap \gamma_t \neq \emptyset$ for all sufficiently small $t > 0$ (note that, by construction, $I$ is nonempty). The proof is completed by showing that $I = \{1, \dots, k\}$ and $\gamma_0 = \cup_{i\in I}(\mathrm{Cl}(W_i) \setminus W_i)$. We argue in two steps.

First, suppose by contradiction that $\gamma_0 \neq \cup_{i\in I}(\mathrm{Cl}(W_i) \setminus W_i)$. Without restriction of generality, we can suppose that $1 \in I$ and $\gamma_0 \cap (\mathrm{Cl}(W_1) \setminus W_1) \neq \mathrm{Cl}(W_1) \setminus W_1$. We consider a point $c \in \mathrm{Cl}(W_1) \setminus W_1$ which does not belong to $\gamma_0$ and an analogous family of ovals $\alpha_t \subset Z_t$ constructed in the same way as $\gamma_t$ but in respect to $c$ and the connected set $W_1$. By (6.4), we conclude that $\alpha_0 \cap \gamma_0$ can only contain points which lie in $\mathrm{Tr}(\Gamma)$. This implies that $\Gamma \setminus \mathrm{Tr}(\Gamma)$ has at least two disconnected components $\gamma_0 \setminus \mathrm{Tr}(\Gamma)$ and $\alpha_0 \setminus \mathrm{Tr}(\Gamma)$, which is in contradiction with the definition of $\mathrm{Tr}(\Gamma)$.

Next, suppose, by contradiction, that $\gamma_0 = \cup_{i\in I}(\mathrm{Cl}(W_i) \setminus W_i)$ but $I \neq \{1, \dots, k\}$. Denote by $\Gamma_0 = \cup_{i\notin I}(\mathrm{Cl}(W_i) \setminus W_i)$. Since the level curve $Z_t$ can only cross points of $\Gamma$ near $a \in \mathrm{Tr}(\Gamma) \cup \mathrm{Gen}(\Gamma)$ (see Lemmas 6.17 and 6.18), we conclude that $\gamma_0 \cap \Gamma_0 \cap \mathrm{Tr}(\Gamma) = \emptyset$. Therefore, $\gamma_0 \cap \Gamma_0 \subset \Gamma \setminus \mathrm{Tr}(\Gamma)$ disconnects $\mathbb{R}^2$, which is again in contradiction with the choice of $\mathrm{Tr}(\Gamma)$.

We conclude the proof by remarking that, since $\gamma_t$ converges to $\Gamma$, for small enough $t > 0$, $\gamma_t$ must be a compact set contained in the interior of $U$. □

After noticing that any compact and regular connected component of a planar algebraic set is a topological circle [54, Theorem 2, p. 180], it follows from Proposition 6.19 that the polynomial family of planar vector fields given by (6.2) has $\Gamma$ as a limit periodic set at $\lambda = 0$. Indeed, it is enough to prove that, for every sufficiently small $t > 0$, the topological circle $\gamma_t \subset \mathbb{R}^2$ given by Proposition 6.19 is a limit cycle set of $X_t$. To show this, it suffices to note that

$$\left(\frac{\partial h}{\partial x_2}+h\frac{\partial h}{\partial x_1}\right)\frac{\partial h}{\partial x_1}+\left(-\frac{\partial h}{\partial x_1}+h\frac{\partial h}{\partial x_2}\right)\frac{\partial h}{\partial x_2}=h\left\|\nabla h\right\|^2$$

and, as a consequence, $Z_t$ is an invariant set containing any periodic orbit of $X_t$. Finally, the fact that $\gamma_t$ is regular guarantees that it is a periodic orbit. This proves the converse of Theorem G (under the extra assumption that $\Gamma$ is compact and generic).

### 6.2.3 Construction of non-generic compact limit periodic sets

Let us now fix a compact connected semialgebraic set $\Gamma \subset \mathbb{R}^2$ with empty interior and $NG(\Gamma) \neq \emptyset$. Denote by $f = f_\Gamma$ the polynomial associated with $\Gamma$, $A_\Gamma$ the associated algebraic set and $S = \mathrm{Gen}(\Gamma) \cup \mathrm{Tr}(\Gamma)$. Fix a coordinate system $x = (x_1, x_2)$ of $\mathbb{R}^2$ and parameters $(\alpha, \lambda) \in \mathbb{R}^{2n_\Gamma+1}$ where $\alpha = (\alpha^1, \ldots, \alpha^n) \in \mathbb{R}^{2n_\Gamma}$. We consider the function

$$h(x,\alpha,\lambda) = f(x)^2 - \lambda \prod_{p\in S}\left(\|x-p\|^2-\lambda^2\right)\prod_{i=1}^{n}\left(\|x-\alpha^i\|^2-\lambda^2\right).$$

Let us take the number $n_\Gamma$, the sequence $(\alpha_i)_{i\in\mathbb{N}}$ and the point $\alpha_0$ as in Remark 6.14 and, for every $i \in \mathbb{N}$, let us consider $h_i(x,\lambda) = h(x,\alpha_i,\lambda)$. For any $i \in \mathbb{N}$, we can apply Proposition 6.19 to the semialgebraic set $\Gamma_i$ introduced in Remark 6.15 to deduce that there exists a value $0 < \lambda_i < \frac{1}{i}$ such that the level set $(h_i(x,\lambda) = 0) \cap (\lambda = \lambda_i)$ contains a subset $\gamma_i$ which is regular connected, compact and $\frac{1}{i}$-close (in respect to the Hausdorff topology) to $\Gamma_i$. Furthermore, apart from shrinking $\lambda_i$ if necessary, we can suppose that $\gamma_i \cap NG(\Gamma) = \emptyset$, because $NG(\Gamma) \subset Alg(\Gamma_i)$ and Lemma 6.17. In particular, note that $\gamma_i \to \Gamma$ when $i \to \infty$ since $\Gamma_i$ converges to $\Gamma$.

*Remark* 6.20. We note that, for every point $a \in \Gamma$, there exists $N > 0$ such that $\gamma_i \cap \{a\} = \emptyset$ for every $i > N$. Indeed, by construction $\gamma_i$ only crosses $\Gamma_i \supset \Gamma$ near the points $\mathrm{Tr}(\Gamma) \cup \mathrm{Gen}(\Gamma_i)$. So, assuming by contradiction that there exists a point $a \in \Gamma$ so that $\gamma_i \cap \{a\} \neq \emptyset$ for an infinite number of $i$, we conclude that $a \in \mathrm{Tr}(\Gamma) \cup \mathrm{Gen}(\Gamma) \cup NG(\Gamma)$. Next, by Lemma 6.18 we conclude that $a \in NG(\Gamma)$, which contradicts the choice of $\lambda_i$.

It follows from the above considerations (just as in the previous Section) that the algebraic family of vector fields

$$X_{\alpha,\lambda} = \left(\frac{\partial h}{\partial x_2}+h\frac{\partial h}{\partial x_1}\right)\partial_{x_1}+\left(-\frac{\partial h}{\partial x_1}h+h\frac{\partial h}{\partial x_2}\right)\partial_{x_2}$$

has $\Gamma$ as a limit periodic set at $(\alpha,\lambda) = (\alpha_0, 0)$.

### 6.2.4 Construction of unbounded limit periodic sets

Finally, let $\Gamma \subset \mathbb{R}^2$ be a closed and unbounded semialgebraic set with empty interior whose compactification $\hat{\Gamma}$ is connected. Apart from considering a translation of $\mathbb{R}^2$, we can assume that $(0,0) \notin \Gamma$.

Let us consider the transition homeomorphism associated with the Bendixson compactification $\phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ given by $\phi(x_1, x_2) = (x_1/r, -x_2/r)$ where $r = x_1^2 + x_2^2$ (see Section 1.4.3).

Note that $\phi(\Gamma)$ is a semialgebraic set (by e. g. [14, Corollary 1.8]), whose closure $Z = \phi(\Gamma) \cup \{0\}$ is a compact connected semialgebraic set with empty interior. By the previous Sections, there exist a polynomial family of planar vector fields $(Y_\lambda)_{\lambda\in\Lambda}$ and a parameter $\lambda_0$ such that $Z$ is a limit periodic set for the family $(Y_\lambda)_\lambda$ at $\lambda_0$. We denote by $(z_{\lambda_n})_n$ the sequence of limit cycles of $Y_\lambda$ which converge to $Z$.

Let us now consider the map $\Phi : (\mathbb{R}^2 \setminus \{0\}) \times \Lambda \to \mathbb{R}^2 \times \Lambda$ given by $\Phi(x_1, x_2, \lambda) = (\phi(x_1, x_2), \lambda)$. The pull-back $\Phi^*(Y_\lambda)$ is rational and there exists an integer $d \geq 0$ such that $X_\lambda = (x_1^2 + x_2^2)^d \Phi^*(Y_\lambda)$ is a polynomial family of vector fields. According to Remark 6.20, for every sufficiently big $n$, $z_{\lambda_n}$ does not intersect the origin so $\Phi^{-1}(z_{\lambda_n})$ is itself a limit cycles of $X_{\lambda_n}$. It follows from the construction that $\Gamma$ is a limit periodic set of $(X_\lambda)_\lambda$ at $\lambda_0$.

# Chapter 7

# Existence of minimal flows on nonorientable surfaces

A local flow $\Phi$ on a surface $S$ is said to be *transitive* if there exists an orbit of $\Phi$ dense in $S$ (that is, such that its closure in $S$ equals $S$). When all the orbits are dense, $\Phi$ is called *minimal*. A surface is transitive (respectively minimal) when it admits a transitive (respectively minimal) local flow.

The problem of finding transitive local flows on manifolds has a long tradition (see for instance the bibliography in [75]). Two works contributed to solve the problem for surfaces. Firstly, in 1998, J. C. Benière proved in his Ph.D. Thesis [12] that all noncompact orientable surfaces which are not embeddable in the real euclidean plane possess a minimal flow. Independently, in 1999, G. Soler in his Master Thesis [76] and in [44], with V. Jiménez, characterized all transitive surfaces of finite genus. However, up to our knowledge, the minimality of nonorientable surfaces has not been characterized so far. In the paper [29], with D. Peralta-Salas and G. Soler's collaboration, we closed this gap. We were able to fully characterized the case of nonorientable surfaces of finite genus, while some progress was made in the study of the infinite genus ones. This chapter is devoted to expose the results in that project.

There is nothing to say about the study of compact minimal surfaces. According to the Poincaré-Hopf Index Theorem (see Theorems 1.43 and 1.44), if a compact surface $S$ admits a $C^1$ minimal local flow, the Euler characteristic of $S$ must be zero and either $S$ is the torus $\mathbb{T}_2$ or the Klein bottle $\mathbb{B}_2$. The latter case can be discarded because $\mathbb{B}_2$ does not admit nontrivial recurrent orbits [65, Corollary 2.2.2, p. 27]. On the other hand, $\mathbb{T}_2$ admits in fact an analytic minimal flow because so is the irrational flow.

Thus, it suffices to focus on the study of noncompact surfaces. No subsurfaces $S$ of the sphere, the projective plane or the Klein bottle can possess minimal local flows. The reason is that any such flow could be extended to a (global) flow on the whole sphere, the whole projective plane or the whole Klein bottle (see Section 1.3). But these three compact surfaces do not admit flows with nontrivial recurrent orbits, in fact they are the only three compact surfaces with that property (see e.g. [7, Section 2.2]). The nonorientable compact surface of genus 3 (the torus with a cross-cap) and any of its subsurfaces of genus 3 cannot possess minimal flows either. This result is stated in [7, p. 14] without a proof; for the sake of completeness, we provided a proof in [29, Appendix A].

In the literature, it is possible to find some isolated examples of noncompact nonorientable surfaces of finite genus with a minimal flow. For instance, in 1978 C. Gutierrez constructed a minimal flow on the compact nonorientable surface of genus 4 minus two points [37]. Similarly, it can be checked that, for any positive integer $2+n$, the surface obtained after removing $n$ points from a compact nonorientable surface of genus $2+n$ admits a minimal flow [77].

Instead of talking about minimal flows we could indistinguishably use the notion of minimal vector fields. Given a surface $S$ we may say that a $C^r$ vector field $X$ on $S$ (for some $r \geq 1$) is minimal if its associated local flow is minimal. Our aim in this chapter is to complete the characterization of noncompact surfaces of finite genus which admit a minimal $C^\omega$ vector field and to give an example of a nonorientable surface of infinite genus with the same property.

The following is the first main result of the chapter.

**Theorem H.** *Let $S$ be an orientable noncompact surface of finite genus $g \geq 1$ or a nonorientable noncompact surface of finite genus $g \geq 4$. Then $S$ admits a minimal complete analytic vector field.*

As already mentioned, the case of orientable surfaces in Theorem H was proved by Benière [12]. Nevertheless, in order to make our exposition as self-contained as possible, we also include a proof of that case.

Benière's approach for proving Theorem H in the orientable case relies on a geometrical method for gluing together different foliated elementary models. Once the pieces are glued, one gets a compact surface endowed with a foliation with only one singularity and whose regular leaves are all dense. Such a foliation has the additional property of admitting a transversal circle whose associated Poincaré map is an oriented interval exchange transformation. Our proof follows a kind of opposite path. For proving Theorem H (for both orientable and nonorientable surfaces), our approach consists in building surfaces and vector fields by suspending certain kind of interval exchange transformations. This idea is usually employed in the literature to get flows on surfaces with different properties. As far as this procedure is concerned, the reason why the case of nonorientable surfaces has remained unsolved up today has to do with the enormous difficulty involved in the task of constructing nonorientable minimal interval exchange transformations. Accordingly, the keystone of the proof of Theorem H is a recent work by A. Linero and G. Soler which fully characterizes nonorientable interval exchange transformations all whose orbits are dense (see Theorem 7.5 below). Suspending an appropriate nonorientable exchange transformation for every finite $4 \leq g < \infty$, we get a minimal flow on the noncompact surface obtained from the nonorientable compact surface of genus $g$ after removing one point. Nonetheless, in order to achieve minimal flows on any nonorientable noncompact surface of genus $g$, additional nontrivial work is still needed: one has to remove a Cantor set of points from the compact surface in such a way that the restricted flow is still minimal. This task follows similar ideas to those presented in Benière's work.

Generalizing his geometrical method, Benière also established the minimality for orientable surfaces of infinite genus. When dealing with nonorientable surfaces of infinite genus, we can prove the following result.

**Theorem I.** *There exist nonorientable surfaces of infinite genus which posseses minimal complete analytic vector fields.*

We emphasize that the proof of Theorem I is independent of the aforementioned Theorem 7.5: the construction of such a minimal vector field is obtained by applying the suspension method to a minimal interval exchange transformation with infinitely many discontinuities. In particular, we prove that:

**Proposition 7.1.** *There exists a minimal interval exchange transformation with flips and with infinitely many points of discontinuity.*

It is worth pointing out that, as far as we know, such an example is new in the literature. We conjecture that a future development in the study of interval exchange transformations with infinitely many discontinuities will allow to prove that any nonorientable surface of infinite genus is minimal.

The content of the chapter is organized as follows. In Section 7.1 we introduce the notion of generalized interval exchange transformation and summarize some related results. The proof of Proposition 7.1 is presented in Section 7.2. In Section 7.3 we explain how to construct minimal vector fields by the suspension of interval exchange transformations. Finally, Theorems H and I are proved in Section 7.4.

## 7.1 Generalized interval exchange transformations

Let $a < b$ be two real numbers and $D$ be an open subset of $(a, b)$. The connected components of $D$ form a countable family of open intervals of $\mathbb{R}$; that is, there are some

$n \in \mathbb{N} \cup \{\infty\}$ and a family of pairwise disjoint open intervals $\{I_i = (a_i, a_{i+1})\}_{i \in \mathbb{N}_n}$ with $D = \cup_{i \in \mathbb{N}_n} I_i$ (if $n = \infty$, we are admitting here degenerated intervals $(a_i, a_{i+1})$ with $a_i = a_{i+1}$ for some indexes $i$, that is, we are considering the empty set as an open interval). Following [39] we say that an injective map $T : D \to [a, b]$ is a *generalized interval exchange transformation in* $(a, b)$, abbreviated as g.i.e.t., if both $D$ and its image $T(D)$ are open and dense subsets of $(a, b)$ and $T$ homeomorphically takes each connected component of $D$ onto a connected component of $T(D)$.

We will focus only on the family of g.i.e.t.'s $T : D \to [a, b]$ with the extra property that the restriction of $T$ to any of the components of $D$ is an affine map of constant slope equal to 1 or $-1$; given a g.i.e.t. $T$ of such a family we will say that it is an *interval exchange transformation (of $n$-intervals)*, abbreviate as *$n$-i.e.t.*

In Subsection 7.1.2 we will analyse some properties of $n$-i.e.t.'s with $n \in \mathbb{N}$, while the case of $\infty$-i.e.t.'s is relegated to Section 7.2; beforehand, in Subsection 7.1.1, we present some definitions equally valid for both cases.

### 7.1.1 Definitions

Let $T : D \to [a, b]$ be an $n$-i.e.t. ($n \in \mathbb{N}$ or $n = \infty$) and $\{I_i = (a_i, a_{i+1})\}_{i \in \mathbb{N}_n}$ be the collection of the connected components of $D$. Observe that $T^{-1}$ is also an $n$-i.e.t. The points in $\{a_i\}_{i \in \mathbb{N}_{n+1}}$ are called the *discontinuities of* $T$. A discontinuity $a_i \notin \{a, b\}$ is said to be a *false discontinuity* if $\lim_{x \to a_i^+} T(x) = \lim_{x \to a_i^-} T(x)$. In the absence of false discontinuities we say that $T$ is a *proper* $n$-i.e.t., in whose case $T^{-1}$ is also proper. If $T$ *reverses the orientation* in some of the interval $I_i$ (i.e. the slope is $-1$ in that interval) we say $T$ is an $n$-i.e.t. with flips; otherwise we can say that $T$ is an i.e.t. *without flips* or an *oriented* i.e.t. When $T$ reverses the orientation exactly in $k$ components of $D$, we remark it by saying that $T$ is an interval exchange transformation of $n$-intervals with $k$-flips or, simply, *an (n,k)-i.e.t.*

If we replace $[a, b]$ by $\mathbb{S}^1 = [a, b]/\equiv$, (where $a \equiv b$), we receive the notion of *circle exchange transformation of $n$-intervals*, abbreviated as *$n$-c.e.t.* Given an $n$-i.e.t. $T : D \to [a, b]$ as above, we will denote as $T^c$ the $n$-c.e.t. obtained after identifying $a$ and $b$. The meaning of the notions of $n$-c.e.t.'s (respectively g.c.e.t.'s) with flips and of $(n, k)$*-c.e.t.'s* are obvious; same comment works for the concepts of false discontinuities and of properness. When working with $\mathbb{S}^1 = [a, b]/\equiv$, and for the sake of simplicity, given any $x, y \in [a, b]$ we will still name them as $x$ and $y$ seen as points in $\mathbb{S}^1$ (with the only precaution that $a = b$ in $\mathbb{S}^1$). Given two points $x < y$ in $[a, b]$ (which are different when they are seen in $\mathbb{S}^1$), the set $\mathbb{S}^1 \setminus \{x, y\}$ possesses two components, two open arcs: one of them is exactly the interval $(x, y) \subset [a, b]$ seen in $\mathbb{S}^1$ (under the convention above), the other one will be denoted as $(y, x)$ (this corresponds with the points $[a, x) \cup (y, b] \subset [a, b]$).

Let $T$ be an $n$-i.e.t. with $n \in \mathbb{N} \cup \{\infty\}$(respectively an $n$-c.e.t.). Let $x \in (a, b)$

(respectively $x \in \mathbb{S}^1$) then the forward (respectively backward) orbit of $x$ generated by $T$ is the set $\mathcal{O}_T^+(x) = \{T^m(x) : m \in \mathbb{N} \cup \{0\}$ and $T^m(x)$ is defined$\}$ (respectively $\mathcal{O}_T^-(x) = \{T^{-m}(x) : m \in \mathbb{N} \cup \{0\}$ and $T^{-m}(x)$ is defined$\}$). The *orbit* of $x$ generated by $T$ is $\mathcal{O}_T(x) = \mathcal{O}_T^-(x) \cup \mathcal{O}_T^+(x)$. Moreover, reducing in this sentence only to case of $T$ being an i.e.t., we define $\mathcal{O}_T(a) = \{a\} \cup \mathcal{O}_T(\lim_{x\to a^+} T(x))$ and $\mathcal{O}_T(b) = \{b\} \cup \mathcal{O}_T(\lim_{x\to b^-} T(x))$. We say that $T$ is *minimal* (respectively *transitive*) if for any $x \in [a,b]$ (respectively if for some $x \in [a,b]$), $\mathcal{O}_T(x)$ is dense in $[a,b]$; this implicitly means that, in particular, $x$ has either a *full forward orbit* ($T^n(x)$ is defined for any $n \geq 0$) or a *full backward orbit* ($T^n(x)$ is defined for any $n \leq 0$). A point $x \in (a,b)$ is said to have *full orbit* if it has full backward and forward orbit.

### 7.1.2 Minimal interval exchange transformations

For any pair $(n,k) \in \mathbb{N}^2$ with $1 \leq k \leq n$ and $n+k \leq 4$, there are no minimal $(n,k)$-i.e.t. (in fact there are no transitive $(n,k)$-c.e.t., as Gutierrez et al. proved in [40]). For all the rest of the pairs $(n,k)$ with $n \in \mathbb{N}$ and $1 \leq k \leq n$ it is always possible to consider a minimal $(n,k)$-i.e.t. The role of this subsection is to clarify these claims.

Here and subsequently, when working with an $n$-i.e.t. $T : D \to [a,b]$, for some $n \in \mathbb{N}$, with $D \subset (a,b)$ having as connected components the open intervals $\{I_i = (a_i, a_{i+1})\}_{1\leq i\leq n}$ we will always assume that $a = a_1 < a_2 < \cdots < a_{n+1} = b$.

We will write $T(a_i^{\oplus}) = \lim_{x\to a_i^+} T(x)$ for $1 \leq i \leq n$ and $T(a_i^{\ominus}) = \lim_{x\to a_i^-} T(x)$ for $2 \leq i \leq n+1$. We also write $T(a_1^{\ominus}) = T(a_1^{\oplus})$ and $T(a_{n+1}^{\oplus}) = T(a_{n+1}^{\ominus})$. A *saddle connection* for $T$ is a set $\mathcal{S} = \{a_i, T(a_i^{\otimes}), \ldots, T^k(a_i^{\otimes}) = a_j\}$ with $k \geq 1$, $\otimes \in \{\oplus, \ominus\}$, $\mathcal{S} \cap \{a_r\}_{r=1}^{n+1} = \{a_i, a_j\}$ and possibly $i = j$. Observe that any *i.e.t.* has saddle connections with $j \in \{1, n+1\}$ and $\mathrm{Card}(\mathcal{S}) = 1$ or 2, these are called *trivial saddle connections*.

*Remark* 7.2. When $T$ is minimal it is obvious that it has no nontrivial saddle connections (by definition of minimality we have, for every $2 \leq j \leq n$, $\mathcal{O}_T(a_j) = \mathcal{O}_T^-(a_j)$ is dense and therefore infinite in particular). Conversely, if $T$ has no nontrivial saddle connections, then $T$ is minimal and in fact any forward or backward orbit through any point is dense when it exists (see [48, Corollary 14.5.12]). It is important to stress that in the statement of [48, Corollary 14.5.12] the hypothesis on the absence of saddle connections refers to the absence of nontrivial saddle connections.

There is a natural injection between the set of $n$-i.e.t.'s and $\mathcal{C}_n = \Lambda^n \times S_n^{\sigma}$, where $\Lambda^n = (0,+\infty)^n$ and $S_n^{\sigma}$ is the set of *(signed) permutations*, where by a permutation we mean an injective map, $\pi : \mathbb{N}_n = \{1,2,\ldots,n\} \to \mathbb{N}_n^{\sigma} = \{-n, -(n-1), \ldots, -1, 1, 2, \ldots, n\}$, such that $|\pi| : \mathbb{N}_n \to \mathbb{N}_n$ is bijective. A signed permutation $\pi$ is said to be a *non-standard permutation* if it is such that for some $i$ it holds $\pi(i) < 0$ (otherwise, $\pi$ is simply a *standard permutation*). As in the case of standard permutations, $\pi$ will be represented by the ordered $n$-tuple $(\pi(1), \pi(2), \ldots, \pi(n))$. We claim that $n$-i.e.t.'s can also be represented with another notation which is more convenient if one needs to use the Rauzy-Veech operator, see [6, 81]; however, the notation we follow is more intuitive for our purposes and agrees with the one followed in [40].

Let $T$ be an $n$-i.e.t. like above, then its associated coordinates in $\mathcal{C}_n$ are $(\lambda, \pi)$ where $\lambda = (\lambda_i)_i$ with $\lambda_i = a_{i+1} - a_i$ for all $i \in \mathbb{N}_n$ and with $\pi(i)$ being positive (respectively negative) if $T|_{I_i}$ has slope 1 (respectively $-1$) and such that, if we order the set $\{T(I_i)\}_{i=1}^n$ in a $n$-tuple taking into account the usual order in $\mathbb{R}$, then $|\pi(i)|$ is the position of the interval $T(I_i)$ in that $n$-tuple.

Conversely, given any $(\lambda, \pi) \in \Lambda^n \times S_n^\sigma$ we can associate it an $n$-i.e.t., $T : \cup_{i=1}^n I_i \subset [0, b] \to [0, b]$, where $b = \sum_{i=1}^n \lambda_i$, $I_1 = (0, \lambda_1)$, $I_i = (\sum_{j=1}^{i-1} \lambda_j, \sum_{j=1}^{i} \lambda_j)$ for any $1 < i \leq n$ and

$$T|_{I_i}(x) = \left( \sum_{j=1}^{|\pi|(i) - \frac{\sigma(\pi(i))+1}{2}} \lambda_{|\pi|^{-1}(j)} \right) + \sigma(\pi(i)) \left[ x - \left( \sum_{j=1}^{i-1} \lambda_j \right) \right] \tag{7.1}$$

for any $1 \leq i \leq n$, where $\sigma(z) = \frac{z}{|z|}$ (the sign of $z$).

These coordinates allow us to make the identification $T \equiv (\lambda, \pi)$.

Notice that if $T \equiv (\lambda, \pi)$ then $T^{-1} \equiv (\mu, \tau)$ with $\tau(j) = \sigma(\pi(|\pi|^{-1}(j)))|\pi|^{-1}(j)$ and $\mu_j = \lambda_{|\pi|^{-1}}(j)$. Combining this fact with Equation (7.1) we see that, for any $m \in \mathbb{Z}$, if $x$ is in the domain of $T^m$ then

$$T^m(x) = \sigma(m)x + k_1(m)\lambda_1 + \cdots + k_n(m)\lambda_n \tag{7.2}$$

with $\sigma(m) \in \{-1, 1\}$ and for certain $k_1(m), \ldots, k_n(m) \in \mathbb{Z}$ (all depending on $x$).

Minimal i.e.t.'s and c.e.t.'s without flips were characterized many years ago by M. Keane (see [49]). Let $T$ be an $n$-i.e.t. in $(a, b)$ without flips and with domain $D = \bigcup_{i=1}^n (a_i, a_{i+1})$. Let $\overline{T}$ be the right continuous extension of $T$ to $[a, b)$. Then, we say that $T$ satisfies the *Keane condition* if $\overline{T}^m(a_i) \neq a_j$ for all $m \geq 1$, $1 \leq i, j \leq n$ and $j \neq 1$.

**Theorem 7.3** (Keane). *Let $T$ be an oriented $n$-i.e.t. that satisfies the Keane condition, then $T$ is minimal.*

A permutation $\pi : \mathbb{N}_n \to \mathbb{N}_n^\sigma$ is called *irreducible* if $|\pi(\{1, 2, \ldots, l\})| \neq \{1, 2, \ldots, l\}$ for any $1 \leq l < n$. Naturally, if $T \equiv (\lambda, \pi)$ is a minimal i.e.t. then $\pi$ must be irreducible. On the other hand, when $T$ has no flips and the components of $\lambda$ are rationally independent, the converse is also true. We formalize this statement in the following lemma (stated and proved in [49]).

**Lemma 7.4.** *If $T \equiv (\lambda, \pi)$ is an i.e.t. without flips, $\pi$ is irreducible and the components of $\lambda$ are rationally independent, $T$ satisfies the Keane condition and hence it is minimal.*

For the case of i.e.t.'s with flips, A. Linero and G. Soler have recently obtained [58] the following result, which will play an essential role in this chapter.

**Theorem 7.5** (Linero and Soler)**.** *Let $n \geq 4$ and $1 \leq k \leq n$. Then there exists a proper, minimal and uniquely ergodic $(n,k)$-i.e.t.*

*Remark* 7.6. A proper minimal $(n,k)$-i.e.t. in $(a,b)$, $T$, always generate a minimal $(n,k)$-c.e.t., $T^c$, after identifying $a$ and $b$. However the second does not have to be proper if we receive a false discontinuity in $a \equiv b$; this occurs if for some $1 \leq i \leq n$, either $T$ preserves the orientation in both $I_i$ and $I_{i+1}$, $\lim_{x \to a_{i+1}^-} T(x) = b$ and $\lim_{x \to a_{i+1}^+} T(x) = a$ (in this case $T^c$ is a proper minimal $(n-1,k)$-c.e.t.) or $T$ reverses the orientation in both $I_i$ and $I_{i+1}$, $\lim_{x \to a_{i+1}^-} T(x) = a$ and $\lim_{x \to a_{i+1}^+} T(x) = b$ (here $T^c$ is a proper minimal $(n-1,k-1)$-c.e.t.). Here, we are calling $I_{n+1} = I_1$. In other words, a proper minimal $(n,k)$-*i.e.t.*, $T$, with coordinates $(\lambda,\pi)$, produces a proper minimal $(n-1,k)$-c.e.t. (respectively a minimal $(n-1,k-1)$-c.e.t.) only if, after calling $\pi(n+1) = \pi(1)$, we have $\pi(i) = n$ and $\pi(i+1) = 1$ (respectively $\pi(i) = -1$ and $\pi(i+1) = -n$) for some $1 \leq i \leq n$. So if $T$ is proper and we suppose $T^c$ has been extended by continuity, there exists $n' \in \{n-1,n\}$ and points $\{c_i\}_{i \in \mathbb{N}_{n'+1}} \subset \{a_i\}_{i \in \mathbb{N}_{n+1}}$ such that $T^c$ is a proper $n'$-c.e.t. which exchanges the intervals $(c_i, c_{i+1})$.

The i.e.t. in $(a,b)$ given by the previous result in the case $k = n-2$ can in fact be taken with the extra property of obtaining a $(n-1,k)$-c.e.t. after identifying $a$ and $b$ in $[a,b]$. Indeed, in [58] the authors build a minimal proper and uniquely ergodic $(n,n-2)$-i.e.t., $T \equiv (\lambda,\pi)$, with $\pi = (-3,-4,-5,\ldots,-[n-1],n,1,-2)$.

## 7.2 Infinite interval exchange transformations: proof of Proposition 7.1

Despite the fact that there are some examples of g.i.e.t.'s in the literature, see for example Chacon transformations in [17] and the interesting way of modifying g.i.e.t.'s analysed by Gutierrez et al. in [39], we have not found such examples when dealing with $\infty$-i.e.t.'s with flips. We dedicate this section to fill this gap.

For the sake of clarity, we divide our exposition in two subsections. In Subsection 7.2.1, we present a procedure for building new minimal i.e.t.'s modifying the definition of a given i.e.t. in certain interval. In Subsection 7.2.2, we iterate that method to construct examples of $\infty$-i.e.t.'s with flips and, in particular, to prove Proposition 7.1.

### 7.2.1 Modifying minimal i.e.t.'s in intervals

Let us consider a proper $(n,k)$-i.e.t. in $(0,1)$, $T : D = \cup_{i=1}^{n}(a_i,a_{i+1}) \to [0,1]$, for some $1 \leq k \leq n < \infty$ and an interval $\Delta \subset [0,1]$.

A direct application of the well-known *Poincaré Recurrence Theorem* (see, for example, [48, Theorem 4.1.19, p. 142]) shows that the set $D_\Delta = \{x \in \Delta : T^m(x) \in \Delta \text{ for some } m \in \mathbb{N}\}$ contains almost every point of $\Delta$ (i.e. $\Delta \setminus D_\Delta$ has zero Lebesgue measure). Associated with this $D_\Delta$, we may define the *Poincaré map* (or the *first return map*) of $T$ on $\Delta$ as the map $T_\Delta : D_\Delta \to \Delta$ which takes every point $x \in D_\Delta$ to $T_\Delta(x) = T^{m_x}(x)$ where $m_x = \min\{m \in \mathbb{N} : T^m(x) \in \Delta\}$. As it is proved in [48, Lemma 14.5.7], $T_\Delta$ determines a proper $(n_\Delta, k_\Delta)$-i.e.t. in the interval $\mathrm{Int}(\Delta)$ with $1 \leq k_\Delta \leq n_\Delta \leq n+2$.

*Remark* 7.7. A precise examination of the proof of [48, Lemma 14.5.7] shows that if $c \in \mathrm{Int}\,\Delta$ is a point of discontinuity of $T_\Delta$, then there exists $m \in \mathbb{N}$ such that $\{T^l(c)\}_{l=1}^{m-1} \cap \Delta = \emptyset$ and either $T^m(c) = a_j$ for some $j \in \{2, \ldots, n\}$ or $T^m(c) \in \partial\Delta$.

Combining $T$ and $T_\Delta$ we are now able to consider a new i.e.t. in $(0,1)$ with more discontinuities than $T$.

**Definition 7.8.** We call $D_\Delta^* = (D \setminus \Delta) \,\dot\cup\, D_\Delta$ and consider the map $T_\Delta^* : D_\Delta^* \to [0,1]$ given by

$$T_\Delta^*(x) = \begin{cases} T(x)\,, & \text{if } x \in D \setminus \Delta, \\ (T \circ T_\Delta)(x)\,, & \text{if } x \in D_\Delta. \end{cases}$$

It follows directly from the definition, that $T_\Delta^*$ gives an i.e.t. in $(0,1)$ which has as discontinuity set the union of the discontinuity sets of $T$, $\{a_i\}_{i=1}^{n+1}$, and of $T_\Delta$, $\{c_i\}_{i=1}^{n_\Delta+1}$. Also, for every $x \in [0,1]$, $\mathcal{O}_{T_\Delta^*}(x) \subset \mathcal{O}_T(x)$.

**Lemma 7.9.** *If in the procedure above we suppose that $T$ is minimal, that $\Delta = (d,f)$ does not contain discontinuity points of $T$, that*

$$\partial\Delta \cap \bigcup_{i=1}^{n+1} \left(\mathcal{O}_T(a_i) \cup \mathcal{O}_T(T(a_i^{\oplus})) \cup \mathcal{O}_T(T(a_i^{\ominus}))\right) = \emptyset \tag{7.3}$$

*and that*

$$\mathcal{O}_T(d) \cap \mathcal{O}_T(f) = \emptyset, \tag{7.4}$$

*then $T_\Delta^*$ is a minimal $(n_\Delta^*, k_\Delta^*)$-iet with $n_\Delta^* = 2n+3$ and $k_\Delta^* \geq 1$.*

*Proof.* According to Remark 7.2, showing the non existence of nontrivial saddle connections for $T_\Delta^*$ is sufficient to guaranteeing its minimality. Let us proceed by contradiction and let $\mathcal{S}$ be such a nontrivial saddle connection. There exists $k \in \mathbb{N}$ and $\otimes \in \{\ominus, \oplus\}$ such that $\mathcal{S}$ is in of one of the following four cases.

**Case 1.** $\mathcal{S} = \{a_i, T_\Delta^*(a_i^{\otimes}), \ldots, (T_\Delta^*)^k(a_i^{\otimes}) = a_j\}$ with $1 \leq i, j \leq n+1$. In this case $\mathcal{S}$ is clearly contained in a nontrivial saddle connection of $T$ which contradicts its minimality.

**Case 2.** $\mathcal{S} = \{a_i, T_\Delta^*(a_i^{\otimes}), \ldots, (T_\Delta^*)^k(a_i^{\otimes}) = c_j\}$ with $1 \leq i \leq n+1$ and $1 \leq j \leq n_\Delta + 1$. Observe that, in light of (7.3), $j \notin \{1, n_\Delta + 1\}$. Remark 7.7 and again (7.3) imply the existence of $h \in \mathbb{N} \cup \{0\}$ and $2 \leq l \leq n$ for which $T^h(c_j) = a_l$; on the other hand, the equality

$(T_\Delta^*)^k(a_i^\otimes) = c_j$ means in particular that there exists $m \in \mathbb{N}$ such that $T^m(a_i^\otimes) = c_j$. Then we deduce that $\mathcal{S}' = \{a_i, T(a_i^\otimes), \ldots, T^{h+m}(a_i^\otimes) = a_l\}$ is a nontrivial saddle connection of $T$.

**Case 3.** $\mathcal{S} = \{c_i, T_\Delta^*(c_i^\otimes), \ldots, (T_\Delta^*)^k(c_i^\otimes) = a_j\}$ with $1 \le i \le n_\Delta + 1$ and $1 \le j \le n+1$. As before, (7.3) and Remark 7.7 guarantee that $i \notin \{1, n_\Delta + 1\}$ and the existence of $h \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $2 \le l \le n$, satisfying $T^h(c_i) = a_l$ and $T^m(a_l^\oslash) = T_\Delta^*(c_i^\otimes)$ for some $\oslash \in \{\ominus, \oplus\}$. Thus we obtain the existence of $p \in \mathbb{N}$ for which $T^p(a_l^\oslash) = a_j$ and $\mathcal{S}' = \{a_l, T(a_l^\oslash), \ldots, T^p(a_l^\oslash) = a_j\}$ is a nontrivial saddle connection of $T$.

**Case 4.** $\mathcal{S} = \{c_i, T_\Delta^*(c_i^\otimes), \ldots, (T_\Delta^*)^k(c_i^\otimes) = c_j\}$ with $1 \le i, j \le n_\Delta + 1$. First, assume that $\{i, j\} \cap \{1, n_\Delta + 1\} \neq \emptyset$ then several possibilities arises. The first one, $\{i, j\} = \{1, n_\Delta + 1\}$, cannot occur since it contradicts (7.4), then either $i \in \{1, n_\Delta + 1\}$ and $j \notin \{1, n_\Delta + 1\}$ or $j \in \{1, n_\Delta + 1\}$ and $i \notin \{1, n_\Delta + 1\}$. In both cases, reasoning respectively as in the second and third item, we obtain a contradiction with (7.3). Assume now that $2 \le i, j \le n_\Delta$, reasoning in the same manner that in the previous item we can obtain $p \in \mathbb{N}$, $2 \le l \le n$ and $\oslash \in \{\ominus, \oplus\}$ for which $T^p(a_l^\oslash) = c_i$. Now we obtain the existence of $m \in \mathbb{N}$ and $2 \le s \le n$ from Remark 7.7 such that $T^m(c_j) = a_s$. Then $T^{p+m}(a_l^\oslash) = a_s$ which implies again the existence of a nontrivial saddle connection for $T$.

Furthermore, since $T$ is minimal we have that for every $2 \le i \le n$ the backward orbit $\mathcal{O}_T^-(a_i)$ meets the open interval $\Delta$: this produces a minimum of $n-1$ discontinuity points of $T_\Delta$ in $\Delta$. Moreover, conditions (7.3) and (7.4) and the density of the backward orbits of $d$ and $f$ produce two more discontinuity points of $T_\Delta$ different from these $n-1$ previous ones. In total there are exactly $n+1$ discontinuity points of $T_\Delta$ in $\mathrm{Int}(\Delta)$. Since $\Delta \subset (a_i, a_{i+1})$ for some $1 \le i \le n$, and by condition (7.3) we know that in fact it must be $[d, f] \subset I_i$, we can conclude that $n_\Delta^* = 2n + 3$. $\square$

### 7.2.2 Proof of Proposition 7.1

Let us now begin with a proper minimal $(n, k)$-i.e.t. in $(0, 1)$, $T : D = \cup_{i=1}^n (a_i, a_{i+1}) \to [0, 1]$, a fix dense set $\{x_i\}_{i \in \mathbb{N}}$ on $[0, 1]$ and a point $p \in (a_1, a_2)$ with full orbit.

We will build inductively a sequence of i.e.t.'s, $(S_i)_{i \in \mathbb{N} \cup \{0\}}$, whose combination allows us to get an example of a minimal $\infty$-i.e.t. with flips.

We start defining $S_0 = T$.

Given any $i \in \mathbb{N}$, suppose $S_{i-1}$ has been already defined with the property of being a minimal i.e.t. such that $p$ has full orbit under it. Hence, by Remark 7.2, there must exist a minimal natural $n_i$ such that for every $1 \le h \le i$ both $\{S_{i-1}^j(p)\}_{j=0}^{n_i}$ and $\{S_{i-1}^j(p)\}_{j=-n_i}^{0}$ meet $\left(x_h - \frac{1}{i}, x_h + \frac{1}{i}\right)$. We then define $S_i = (S_{i-1})_{\Delta_i}^*$ where $\Delta_i = (d_i, f_i) \subset (p, a_2)$ is such that

C-1. $|\Delta_i| = f_i - d_i < d_i - p$ and, when $i \ge 2$, $f_i < d_{i-1}$;

C-2. $f_i - p < 1/i$;

C-3. $\partial\Delta_i \cap \mathcal{O}_T(p) = \emptyset$;

C-4. If $Q_{i-1}$ is the set of discontinuities of $S_{i-1}$, then

$$\partial\Delta_i \bigcap \bigcup_{x\in Q_{i-1}} (\mathcal{O}_{S_{i-1}}(x) \cup \mathcal{O}_{S_{i-1}}(S_{i-1}(x^{\oplus})) \cup \mathcal{O}_{S_{i-1}}(S_{i-1}(x^{\ominus})) = \emptyset,$$

and

$$\mathcal{O}_{S_{i-1}}(d_i) \cap \mathcal{O}_{S_{i-1}}(f_i) = \emptyset;$$

C-5. $\Delta_i \cap \{S_{i-1}^j(p)\}_{j=-n_i}^{n_i} = \emptyset$.

Notice that, because of Lemma 7.9, properties C-1 and C-4 ensure that $S_i$ is also a minimal i.e.t. and, because of property C-3, $p$ has also full orbit for $S_i$.

Let us call $E = \bigcup_{i\in\mathbb{N}} \Delta_i$ and observe that, for every $i \in \mathbb{N}$,

1. $S_{i-1}|_{\Delta_i}$ is continuous;
2. if $x \in (0,1) \setminus E$, then $S_i(x) = T(x)$;
3. if $x \in \Delta_i$, then $S_k(x) = S_i(x)$ for any $k \geq i$.

This allows us to define $S : D_S \to (0,1)$, where $D_S = D \setminus \cup_{i\in\mathbb{N}} Q_i$, by

$$S(x) = \begin{cases} S_i(x) & \text{if } x \in \Delta_i, \\ T(x) & \text{if } x \in D \setminus E. \end{cases}$$

**Proposition 7.10.** *The function $S$ is a minimal $\infty$-i.e.t. with flips.*

*Proof.* The fact that $S$ is an $\infty$-i.e.t. with flips is clear. We prove the minimality of $S$ by stages.

Firstly, for any $x \in (0,1)$ the orbit $\mathcal{O}_S(x)$ is infinite, that is, either $x$ has full forward orbit or full backward orbit. Indeed, if for some $x \in (0,1)$ and some $* \in \{-,+\}$ the set $\mathcal{O}_S^*(x)$ is finite, we may take the maximal natural $i$ such that $\mathcal{O}_S^*(x) \cap \Delta_i \neq \emptyset$ and observe that $\mathcal{O}_S^*(x) = \mathcal{O}_{S_{i+1}}^*(x)$. But, on account of the minimality of $S_{i+1}$, $\mathcal{O}_{S_{i+1}}^-(x)$ and $\mathcal{O}_{S_{i+1}}^+(x)$ cannot be simultaneously finite.

Secondly, $p \in \mathrm{Cl}(\mathcal{O}_S(x))$ for any $x \in (0,1)$. If $p \notin \mathrm{Cl}(\mathcal{O}_S(x))$, then $\mathcal{O}_S(x)$ would only intersect a finite number of intervals $\Delta_i$ (because of C-2) and, as in the previous paragraph, we arrive to a contradiction with the minimality of some $S_k$, $k \in \mathbb{N}$.

Thirdly, $\lim_{x\to p^+} S(x) = T(p)$ and therefore $S$ is continuous on $p$. Indeed, fix $\epsilon > 0$ and observe that $S(x) - T(x) = 0$ if $x \in E^c$ and $|S(x) - T(x)| < |\Delta_i| < x - p$ if $x \in \Delta_i$ (because of C-1). Thus, for a sufficiently small $\delta > 0$ we have that for every $x \in (p-\delta, p+\delta)$, $|S(x) - T(p)| \leq |S(x) - T(x)| + |T(x) - T(p)| < 2|x-p| < \epsilon$.

Fourthly, both the backward and the forward orbits of $p$ generated by $S$ are dense in $[0,1]$. To prove it, it is clearly sufficient to see that, for any $h, i \in \mathbb{N}$ with $h \leq i$ and any $* \in \{-,+\}$, $\mathcal{O}_S^*(p) \cap \left(x_h - \frac{1}{i}, x_h + \frac{1}{i}\right) \neq \emptyset$. But if $h \leq i$ and $* \in \{-,+\}$, we know that $\left\{S_{i-1}^j(p)\right\}_{j=0}^{*n_i} \cap \left(x_h - \frac{1}{i}, x_h + \frac{1}{i}\right) \neq \emptyset$ and that, by C-5, $\left\{S^j(p)\right\}_{j=0}^{*n_i} = \left\{S_{i-1}^j(p)\right\}_{j=0}^{*n_i}$.

Finally, we take $x \in (0,1)$ and we see that for any $c \in (0,1)$ and any $\epsilon > 0$, $\mathcal{O}_S(x) \cap (c-\epsilon, c+\epsilon) \neq \emptyset$. Since $\mathcal{O}_S^*(p)$ is dense (for any $* \in \{-,+\}$) we can take an integer $m_2$ (note that the sign can be chosen as desired) for which $|S^{m_2}(p) - c| < \frac{\epsilon}{2}$. Observe that, because of C-3 and the fact that $S$ is continuous at $p$, $S^{m_2}$ is continuous at $p$. So there exists $\delta > 0$ such that if $|y - p| < \delta$ then $|S^{m_2}(y) - S^{m_2}(p)| < \frac{\epsilon}{2}$. Since $p \in \mathrm{Cl}(\mathcal{O}_S(x))$, there exists also an integer $m_1$ satisfying $|S^{m_1}(x) - p| < \delta$. Thus $|S^{m_1+m_2}(x) - c| \leq |S^{m_1+m_2}(x) - S^{m_2}(p)| + |S^{m_2}(p) - c| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Then $S^{m_1+m_2}(x) \in (c-\epsilon, c+\epsilon)$ as desired. □

## 7.3 Building vector fields from circle exchange transformations

Circle exchange transformations and vector fields are related by means of a standard procedure called *suspension of circle exchange transformations*. In this section we introduce this procedure following [39, Section 6] (with minor changes). We have been also strongly inspired by [8, 57] and [77].

Let $D$ be an open dense subset of $(0,1)$ and $\{I_i = (a_i, a_{i+1})\}_{i \in \mathbb{N}_n}$ be the family of its connected components ($n \in \mathbb{N} \cup \{\infty\}$). Let $T : D \to [0,1]$ be a proper $(n,k)$-i.e.t. (for some $1 \leq k \leq n$). When $n \in \mathbb{N}$, recall the convention of supposing that $0 = a_1 < a_2 < \cdots < a_{n+1} = 1$. Take also the proper $(n',k')$-c.e.t. $T^c : \cup_{i \in \mathbb{N}_{n'}} (c_i, c_{i+1}) \subset \mathbb{S}^1 \to \mathbb{S}^1$ associated with $T$ where $k' \in \{k-1, k\}$ and $n' \in \{n-1, n\}$ (see Remark 7.6). Consider the set of points $\cup_{i \in \mathbb{N}_{n'}} \partial T^c((c_i, c_{i+1}))$ and label them as $\{b_i\}_{i \in \mathbb{N}_{n'+1}} \subset [0,1]$ such as to verify, for every $i, j \in \mathbb{N}_{n'+1}$, $c_i \leq c_j$ if and only if $b_i \leq b_j$. Let $\sigma : \mathbb{N}_{n'} \to \mathbb{N}_{n'}$ be a bijection such that, for every $i \in \mathbb{N}_{n'}$, $T^c((c_i, c_{i+1})) = (b_{\sigma(i)}, b_{\sigma(i)+1})$.

### 7.3.1 The construction of the suspended surface

Figure 7.1 intends to clarify the following construction. We start considering the noncompact $\partial$-surface $N = (\mathbb{S}^1 \times [0,1]) \setminus (P_0 \cup P_1)$ with $P_0 = \{b_i\}_{i \in \mathbb{N}_{n'}} \times \{0\}$ and $P_1 = \{c_i\}_{i \in \mathbb{N}_{n'}} \times \{1\}$.

Call also $\bar{N} = \mathbb{S}^1 \times [0,1]$ and consider both $N$ and $\bar{N}$ as $\partial$-surfaces equipped with the natural analytic structure compatible with their euclidean topological structure (as subsets of $\mathbb{S}^1 \times \mathbb{R}$).

For every $i \in \mathbb{N}_{n'}$, we consider $h_i : [c_i, c_{i+1}] \times \{1\} \to [b_{\sigma(i)}, b_{\sigma(i)+1}] \times \{0\}$ given by the formula $h_i(x, 1) = (T^c(x), 0)$ for every $x \in (c_i, c_{i+1})$ and either $h_i(c_i, 1) = (b_{\sigma(i)}, 0)$ and $h_i(c_{i+1}, 1) = (b_{\sigma(i)+1}, 0)$ (if $T^c$ preserves the orientation in $(c_i, c_{i+1})$) or $h_i(c_{i+1}, 1) = (b_{\sigma(i)}, 0)$ and $h_i(c_i, 1) = (b_{\sigma(i)+1}, 0)$ (if $T^c$ reverses the orientation in $(c_i, c_{i+1})$).

Let $\mathcal{R} \subset \bar{N} \times \bar{N}$ be the smallest equivalence relation of $\bar{N}$ containing all the pairs $(p, q)$ satisfying that, for some $i \in \mathbb{N}_{n'}$, $p \in [c_i, c_{i+1}] \times \{1\}$ and $q = h_i(p)$. Consider the topological quotient $M_T = \bar{N}/\mathcal{R}$ and let $\rho : \bar{N} \to M_T$ be the associated natural projection. When $(x, y) \in \mathcal{R}$ we will also write $x \sim y$ or $[x] = [y]$.

Since $\bar{N}$ is compact and connected so is $M_T$. The subset of $M_T$ given by $S_T = \rho(N)$ has not only topological structure but it is also an analytic surface with boundary: $S_T$ is nothing else than a set built following the standard process of attaching surfaces along their boundaries. We collect this information in the following result.

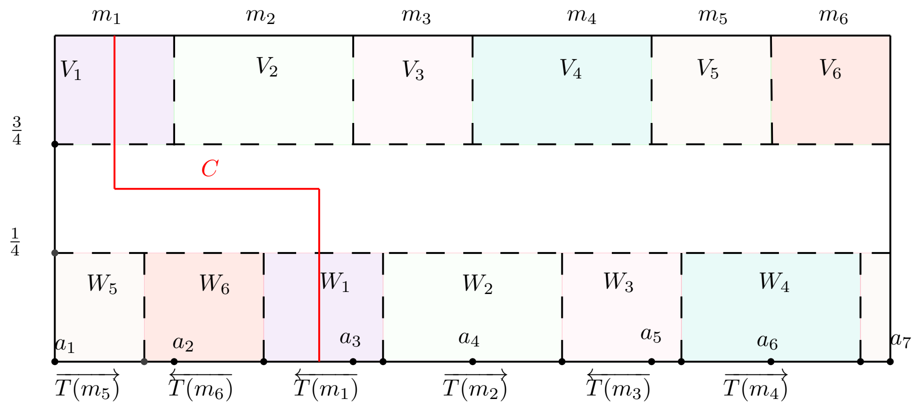


Figure 7.1: Construction of $M_T$ by means of a $(6,3)$-i.e.t. with $\pi = (-3, 4, -5, 6, 1, -2)$. The circle $C$ is nonorientable. The arrows on the images of the $m_i$ mark if they are flipped by $T$.

**Lemma 7.11.** *$S_T$ is an analytic surface. The surface is orientable (respectively nonorientable) when $T^c$ has no flips (respectively has flips). Moreover, if $n \in \mathbb{N}$, $M_T$ is a compact surface which coincides with $S_T$ in genus and in orientability class.*

*Proof.* We start defining, for every index $i \in \mathbb{N}_{n'}$, two open subsets of $N$ as $V_i = (c_i, c_{i+1}) \times (3/4, 1]$ and $W_i = (b_{\sigma(i)}, b_{\sigma(i)+1}) \times [0, 1/4)$ and continuous maps $\phi_i : V_i \cup W_i \to (b_{\sigma(i)}, b_{\sigma(i)+1}) \times (-1/4, 1/4)$ by

$$\phi_i(x,t) = \begin{cases} (T^c(x), t-1), & \text{if } (x,t) \in V_i, \\ (x,t), & \text{if } (x,t) \in W_i. \end{cases} \tag{7.5}$$

Since the restrictions of $\phi_i$ to $V_i$ and to $W_i$ are both embeddings with closed images in $(b_{\sigma(i)}, b_{\sigma(i)+1}) \times (-1/4, 1/4)$, it follows that $\phi_i$ is not only continuous but also closed. Moreover, since $\phi_i$ take the same value in all points $(x, y)$ and $(x', y')$ such that $\rho(x, y) =$

$\rho(x', y')$, we can define a map $\Phi_i : \rho(V_i \cup W_i) \to (b_{\sigma(i)}, b_{\sigma(i)+1}) \times (-1/4, 1/4)$ by $\Phi_i(\rho(x,t)) = \phi_i(x,t)$ for every $(x,t) \in V_i \cup W_i$. This map is bijective and the continuity and closeness of $\phi_i$ guarantees that it is, in fact, an homeomorphism and, in particular, that $\rho(V_i \cup W_i)$ is a surface. On the other hand, the restriction of $\rho$ to $\operatorname{Int} N$ is an embedding and, therefore, every point in $S_T$ possesses an open neighbourhood homeomorphic to an open connected subset of $\mathbb{R}^2$.

The fact that $S_T$ can be rewritten as a countable union of open subsets each of them being second countable shows that $S_T$ is second countable itself: $S_T = \rho(\operatorname{Int} N) \cup \bigcup_{i \in \mathbb{N}_{n'}} \rho(V_i \cup W_i)$. This union also allows us to claim the connectedness of $S_T$: all the sets in the union of the left term in the equality are connected and all of them meet $\rho(\operatorname{Int} N)$. Finally, $S_T$ is clearly Hausdorff. All together shows that $S_T$ is a surface.

Let us denote by $(x, y)$ the components of the identity map on $\operatorname{Int} N$ and by $(x^i, y^i)$ the components of the map $\Phi_i$ for every $i \in \mathbb{N}_{n'}$. As an atlas for $S_T$ we can take the collection of the following coordinates charts $\{(\rho(\operatorname{Int} N), (x, y))\} \cup \{(\rho(V_i \cup W_i), (x^i, y^i))\}_{i \in \mathbb{N}_{n'}}$. It is an immediate computation to verify that the transition maps associated with these coordinate charts are all given by analytic maps and, when $T$ has no flips, have all positive Jacobian determinant everywhere in their domains. Indeed, the only coordinate neighbourhoods with nonempty intersections are the pairs $\{\rho(\operatorname{Int} N), \rho(V_i \cup W_i)\}_i$. Let $i \in \mathbb{N}_{n'}$, we have $\rho(\operatorname{Int} N) \cap \rho(V_i \cup W_i) = \rho(\operatorname{Int} V_i \cup \operatorname{Int} W_i)$. Let $F_i$ be the restriction of $\phi_i = \Phi_i \circ \rho$ to $\operatorname{Int} V_i$ in its domain and to $\phi_i(\operatorname{Int} V_i)$ in its codomain and with formula $F_i(x, y) = \phi_i(x, t)$ where the last term is computed as in equation (7.5). The transition map from $(\rho(\operatorname{Int} N), (x, y))$ to $(\rho(V_i \cup W_i), (x^i, y^i))$ is the map $H_i : \operatorname{Int} V_i \cup \operatorname{Int} W_i \to \phi_i(\operatorname{Int} V_i \cup \operatorname{Int} W_i)$ defined as $H_i(x, y) = F_i(x, y)$ if $(x, y) \in \operatorname{Int} V_i$ and as $H_i(x, y) = \phi_i(x, y) = (x, t)$ otherwise. All these transition maps are therefore analytic diffeomorphisms. Moreover, their Jacobian matrices are trivial. When $T$ preserves the orientation in $(c_i, c_{i+1})$, the map $F_i$ has as Jacobian matrix at any point in its domain the identity; when $T$ reverses the orientation in $(c_i, c_{i+1})$ the Jacobian matrix of $F_i$ at any point $(x, t)$ equals $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. From here we can already conclude that $S_T$ is an orientable surface when $T$ has no flips (in such a case all the transition maps have the the identity as Jacobian matrix so in particular all have positive determinant and hence we have an analytic and consistently oriented atlas for $S_T$). On the other hand, when $T$ has flips, say for example its slope equals $-1$ in $(c_i, c_{i+1})$ and take an open $\Gamma$ arc in $\operatorname{Int} N$ having as endpoints the middle points of $(c_i, c_{i+1}) \times \{1\}$ and $(b_{\sigma(i)}, b_{\sigma(i)+1}) \times \{0\}$, then $\rho(\operatorname{Cl}(\Gamma))$ is clearly a nonorientable circle, see the circle $C$ in Figure 7.1.

Suppose now that we are in the case $n \in \mathbb{N}$. To begin with, a direct examination on the family of open subsets of $M_T$ show that $M_T$ is not only compact and connected but also Hausdorff. Besides, the set $M_T \setminus S_T$ is totally disconnected and nonseparating on $M_T$. Corollary 1.34 together with the Remark 1.35 finish the proof. $\square$

In what follows, we will refer to $S_T$ as the *suspended surface associated with* $T$. When $T$ is an $n$-i.e.t.for some $n \in \mathbb{N}$, $M_T$ will be called the *compact suspended surface associated with* $T$. In this last case, the points in the set $\mathcal{M}_T = \{[(c_i, 1)] : 1 \leq i \leq n'\}$ are called the *marked points of* $M_T$.

### 7.3.2 The construction of the suspended vector field

With the notation introduced in the proof of Lemma 7.11, we can consider a vector field $X$ on $S_T$ defined in local coordinates as

$$X(p) = \begin{cases} \left.\frac{\partial}{\partial y}\right|_p, & \text{if } p \in \rho(\operatorname{Int} N), \\ \left.\frac{\partial}{\partial y^i}\right|_p, & \text{if } p \in \rho(V_i \cup W_i). \end{cases}$$

The trivial shapes which have all the Jacobian matrices associated with the transition maps guarantee that these definitions are consistent (they agree in the non-disjoint coordinate neighbourhoods). Furthermore, $X$ is analytic on the whole $S_T$ because so are its local representatives.

Now, observe that $[\mathbb{S}^1 \times \{1/2\}]$ is a transversal circle to any orbit of $X$ and the Poincaré first return map which is defined over it is given exactly by $T^c$ (when we see $T^c$ as a map from $\mathbb{S}^1 \times \{1/2\}$ to $\mathbb{S}^1 \times \{1/2\}$ after naturally identifying $\mathbb{S}^1 \times \{1/2\}$ with $\mathbb{S}^1$). This simple observation allow us to claim that $X$ is an analytic minimal vector field if and only if $T^c$ is minimal.

Associated with $X$ we can always take an analytic positive map $f_T : S_T \to \mathbb{R}$ such that $X_T = f_T X$ is a complete analytic vector field. We will refer to this analytic vector field $X_T$ as *the suspended vector field associated with* $T$.

## 7.4 Proofs of Theorems H and I

Let us start with the proof of Theorem I, which is immediate from Proposition 7.1. Indeed, Proposition 7.1 gives an infinite minimal g.i.e.t. $T$ with flips, and associated with it we may take the suspended surface $S_T$ and the suspended vector field $X_T$. To conclude, we only need to notice that the surface $S_T$ must be of infinite genus due to the *Structure Theorem* of [38].

In order to deal with the proof of Theorem H, we will restrict now to the case $n \in \mathbb{N}$ in the procedure above. Recall that, since $n \in \mathbb{N}$, the suspended surface $S_T$ is a noncompact surfaced contained in the suspended compact surface $M_T$. The analytic map $f_T : S_T \to \mathbb{R}$ considered above to define the suspended vector field $X_T$ can in fact be taken such that after defining it as zero in the marked points $\mathcal{M}_T = M_T \setminus S_T$ we get a $C^\infty$ map on $M_T$.

Then, the vector field $X_T$ can also be seen as a $C^\infty$ complete vector field on the compact surface $M_T$ whose restriction to $S_T$ is analytic. When $X$ is understood as a vector field on $M_T$ we will refer to it as the *suspended compact vector field associated with* $T$.

Given $p \in M_T$, denote by $\gamma_p : \mathbb{R} \to M_T$ the integral curve of $X_T$ starting at $p$. We already know that for every point $p \in S_T$, the orbit $\Gamma_p = \gamma_p(\mathbb{R})$ is dense in $M_T$. Even more, owing to Remark 7.2, if $p \in [\mathcal{S}^1 \times \{1/2\}]$ there are three possibilities:

1. if $p \notin [\{c_1, \ldots, c_{n'}\} \cup \{b_1, \ldots, b_{n'}\} \times \{1/2\}]$, then both $\Gamma_p^+ = \{\gamma_p(t) : t \in [0, \infty)\}$ and $\Gamma_p^- = \{\gamma_p(t) : t \in (-\infty, 0]\}$ are dense in $M_T$;
2. if $p \in [\{c_1, \ldots, c_{n'}\} \times \{1/2\}]$, $\Gamma_p^+ = [(\{c_i\} \times [1/2, 1)]$ with $\lim_{t \to \infty} \gamma_p(t) = [(c_i, 1)]$ and $\Gamma_p^- = \{\gamma_p(t) : t \in (-\infty, 0]\}$ is dense;
3. if $p \in [\{b_1, \ldots, b_{n'}\} \times \{1/2\}]$, $\Gamma_p^- = [(\{b_i\} \times (0, 1/2])]$ with $\lim_{t \to -\infty} \gamma_p(t) = [(b_i, 0)]$ and $\Gamma_p^+ = \{\gamma_p(t) : t \in [0, \infty)\}$ is dense.

### 7.4.1 Computation of the genus of the surface $M_T$

In order to deduce the genus of the surface $M_T$, we begin computing the index of the singular points of $X_T$. For doing so, it is enough to notice that all the singular points of $X_T$ have neighbourhoods which are topologically equivalent with an open disk on the plane decomposed in evenly many hyperbolic sectors (if in such a decomposition there are exactly $2m$ hyperbolic sectors, we say that the singular point is a $2m$-*saddle point*) — see Figure 7.2.

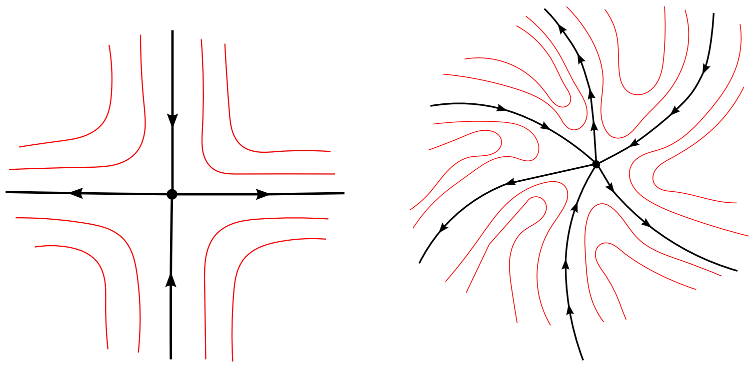

Figure 7.2: A standard saddle point (left) and a 6-saddle point (right).

Any of the marked points $[(c_i, 1)]$, $1 \le i \le n'$, is a $2k$-saddle point for $X_T$ for some $k \ge 2$. Indeed, since $T^c$ is proper, for any $1 \le i \le n'$ we have that $\lim_{x \to c_i^+} T^c(x) \neq \lim_{x \to c_i^-} T^c(x)$ and $\lim_{x \to b_i^+} (T^c)^{-1}(x) \neq \lim_{x \to b_i^-} (T^c)^{-1}(x)$ so the class of equivalence $[(c_i, 1)]$ contains at least some other $[(c_j, 1)]$ ($1 \le j \le n'$) with $c_j \neq c_i$ and as many points of the type $[(c_k, 1)]$ as of the type $[(b_l, 1)]$ ($1 \le k, l \le n'$). Let us say $[(c_i, 1)] =$

$\{(c_{i_1},1),\cdots,(c_{i_k},1)\}\cup\{(b_{j_1},0),\cdots,(b_{j_k},0)\}$ (with both $\{i_1,\cdots,i_k\}$ and $\{j_1,\cdots,j_k\}$ being sets with $k\geq 2$ different points in $\mathbb{N}_{n'}$). For every $1\leq l\leq k$, the semiorbit $\Gamma^+_{[(c_{j_l},1/2)]}$ has $\{[(c_i,1)]\}$ as $\omega$-limit set while the semiorbit $\Gamma^-_{[(b_{j_l},1/2)]}$ has $\{[(c_i,1)]\}$ as $\alpha$-limit set. If $B_i$ is a sufficiently small open ball centred in $[(c_i,1)]$, then it is clear that the orbits of $X_T$ through any point of the semiorbit $\Gamma^+_{[(c_{j_l},1/2)]}$ or of the semiorbit $\Gamma^-_{[(b_{j_l},1/2)]}$, for $1\leq l\leq k$, are the only regular ones meeting $B_i$ and having $[(c_i,1)]$ in one of their limit sets. The rest of the regular orbits $\Gamma$ meeting $B_i$ are such that $\Gamma\cap B_i$ is an open arc with endpoints in the boundary of $B_i$. This produces exactly $2k$ hyperbolic sectors in the decomposition of $B_i$ (Theorem B in Chapter 3 justifies this geometrically clear claim). The index of the point $[(c_i,1)]$ is then exactly $\frac{1}{2}(2-2k)=1-k$.

By the Poincaré-Hopf Index Theorem (see Theorems 1.43 and 1.44), we can already compute the Euler characteristic of the surface $S_T$. Let us say that $M_T$ presents $m\geq 1$ marked points, $[(c_{\iota_1},1)],\ldots,[(c_{\iota_m},1)]$, and that for every $1\leq l\leq m$, $[(c_{\iota_l},1)]$ is a $2k_l$-saddle point. Then then $\chi(S_T)=\sum_{l=1}^{m}(1-k_l)=m-n$. Consequently, if $S_T$ is orientable (respectively nonorientable) its genus equals

$$g(S_T)=1+\frac{n-m}{2}\ \text{(respectively } g(S_T)=2+n-m). \tag{7.6}$$

### 7.4.2 An intermediate result

In this subsection we prove a special case of Theorem H, which plays a key role in the next subsection when proving the theorem in its full generality.

**Theorem 7.12.** *For any $n\geq 4$ there exists a proper minimal $(n,n-2)$-i.e.t., $T$, which produces, after identifying $0$ and $1$, a $(n-1,n-2)$-c.e.t., $T^c$. The associated suspended compact surface, $M_T$, is a nonorientable compact surface of genus $n$ and the suspended compact vector field, $X_T$, is of class $C^\infty$ and has exactly one singular point $p_0$ (with a neighbourhood decomposed in $h=2n-2$ hyperbolic sectors). The restriction of $X_T$ to the suspended surface $S_T=M_T\setminus\{p_0\}$ is analytic and minimal. With more detail, given any point in $S_T$ if its orbit is not both backwardly and forwardly dense, then either it has $\{p_0\}$ as $\alpha$-limit set and is forwardly dense or has $\{p_0\}$ as $\omega$-limit set and is backwardly dense (these last two cases arising only for exactly $h$ orbits).*

*Proof.* Theorem 7.5 and Remark 7.6 provide, for $n\geq 4$, a proper, minimal uniquely ergodic $(n,n-2)$-i.e.t., $T=(\lambda,\pi)$, with $\pi=(-3,-4,-5,\ldots,-[n-1],n,1,-2)$. Denote by $(a_i,a_{i+1})$, $1\leq i\leq n$, the intervals exchanged by $T$. Take the minimal $(n-1,n-2)$-c.e.t., $T^c$, obtained after identifying 0 with 1 and write $(c_i,c_{i+1})$, $1\leq i\leq n-1$, to denote the intervals exchanged by $T^c$. Then we have $c_1=a_1<c_2=a_2<\cdots<c_{n-2}=a_{n-2}<a_{n-1}<c_{n-1}=a_n<c_n=a_{n+1}=1$.

Next we show that the number of marked points appearing in the compact surface $M_T$ in the construction of the suspension is exactly one (see Section 7.3.1). By the construction of $M_T$ it is clear that the permutation $\pi$ gives the identifications $(c_i, 1) \sim (c_{i+2}, 1)$ for any $1 \leq i \leq n-4$. Furthermore, $\pi$ also gives the relations $(c_{n-3}, 1) \sim (c_{n-2}, 1)$, $(c_{n-1}, 1) \sim (c_1, 1)$ and $(c_2, 1) \sim (c_{n-1}, 1)$.

Finally, from Equation (7.6), we deduce that $g(S_{T^c}) = n$. □

*Remark* 7.13. The same argument can be used to guarantee that, considering appropriate *c.e.t.* without flips, the statement of the result works analogously for compact orientable surfaces of genus $g \geq 1$. Indeed, for any $n \in \mathbb{N} \cup \{0\}$ we may take the (standard) irreducible permutation $\pi$ given by $\pi(i) = 2i$ for every $1 \leq i \leq n+1$ and $\pi(i) = 2(i-n+2)+1$ for every $n+2 \leq i \leq 2n+2$: $\pi = (2, 4, \ldots, 2n, 2n+2, 1, 3, \ldots, 2n-1, 2n+1)$.

Consider also a vector $\lambda = (\lambda_i)_{1 \leq i \leq 2n+2}$ with its components being rationally independent. Then $T \equiv (\lambda, \pi)$ satisfies the Keane condition and it is minimal (see Lemma 7.4). After identifying 0 and 1 we get a proper minimal $(2n+1)$-c.e.t., $T^c$. This oriented c.e.t. produces also a unique boundary component when we suspend it and equation (7.6) says now that $g(S_{T^c}) = 1 + \frac{(2n+1)-1}{2} = n+1$.

### 7.4.3 Completing the proof of Theorem H

To complete the proof of Theorem H, we still need to present a couple final technical lemmas regarding Cantor sets.

A nonempty topological space $K$ is said to be a *Cantor set* if it is metrizable, compact, totally disconnected and perfect (i.e. without isolated points). All the Cantor sets are homeomorphic; in particular every Cantor set is homeomorphic to the ternary Cantor set in $\mathbb{R}$ . Even more:

**Theorem 7.14.** *Let $K$ be a Cantor set and fix a point $p_0 \in K$. Then, for any compact, metric, totally disconnected space $L$ there exists an embedding $h : L \to K$ with $p_0 \in h(L)$.*

*Proof.* This is an elementary consequence of a well-known topological result stating that any compact metric totally disconnected space has a homeomorphic copy inside any Cantor set (see [53, p. 285]). Indeed, if $L$ is itself a Cantor set there is nothing to say: two any Cantor sets are homeomorphic. In the contrary case, $L$ has isolated points; let us fix one of these isolated points $q_0 \in L$. Let $h : L \to K$ be an embedding and suppose that $p_0 \notin h(L)$. It is enough then to consider the map $\tilde{h} : L \to K$ given by $\tilde{h}(q) = h(q)$ if $q \neq q_0$ and $\tilde{h}(q_0) = p_0$. This new map is also injective (because so is $h$), continuous (because $q_0$ is isolated) and closed (all the continuous maps from compact spaces to Hausdorff spaces are closed). □

The following result is cited in [12, pp. 14–15], we give a detailed proof. A subset $K \subset \mathbb{R}$ is called *rationally independent* if for any nonempty finite subset $F = \{x_1, \dots, x_k\} \subset K$ there are no integers $n_1, \dots, n_k$ such that $n_1 x_1 + \dots + n_k x_k \in \mathbb{Q}$ and $(n_1)^2 + \dots + (n_k)^2 \neq 0$ (i.e. such that they do not vanish simultaneously); $K$ is said *rationally dependent* otherwise.

**Lemma 7.15.** *Let $I \subset \mathbb{R}$ be an open interval. Any rationally independent finite set $F \subset I$ is contained in a rationally independent Cantor set $K \subset I$.*

*Proof.* Let $I$ be an open interval and $F \subset I$ be finite and rationally independent. We will construct, by induction, a nested sequence of compact sets whose intersection gives a Cantor set $K$ such that $F \subset K \subset I$.

Given any $p_1, \dots, p_j \in \mathbb{Z}$ we name $f_{p_1,\dots,p_j}$ the polynomial which, for every $y_1, \dots, y_j \in \mathbb{R}$, is defined by $f_{p_1,\dots,p_j}(y_1, \dots, y_j) := \sum_{i=1}^{j} p_i y_i$.

If $G \subset \mathbb{R}$ is a finite subset (say $G$ has exactly $m$ elements), the set $H = \{\sum_{i=1}^{m} p_i y_i : p_i \in \mathbb{Z},\, y_i \in G\}$ is infinite. This trivial observation implies that, when $G$ is rationally independent, all but countably many elements of every connected component of $\mathbb{R} \setminus H$ can be added to $G$ and generate a new finite rationally independent set.

Therefore, we can assume we have a finite sequence $a_1^0 < a_2^0 < a_3^0 < \cdots a_{2m-1}^0 < a_{2m}^0$ (in $I$), for some positive integer $m$, such that $F' = \{a_i^0 : 1 \le i \le 2m\}$ is rationally independent and $F = \{a_{2i}^0 : 1 \le i \le m\}$. Call $K_0 = \cup_{i=1}^{m} B_i^0$ with $B_i^0 = [a_{2i-1}^0, a_{2i}^0]$ $(1 \le i \le m)$.

Since $F'$ is rationally independent, for every choice of integers $p_1, \dots, p_{2m}$ in $\mathbb{Z}$ which do not vanish simultaneously, $f_{p_1,\dots,p_{2m}}(a_1^0, \dots, a_{2m}^0) \notin \mathbb{Z}$. By continuity we can then guarantee the existence of $4m$ real numbers, $a_1^1 < \cdots < a_{4m}^1$, with $a_{4i-3}^1 = a_{2i-1}^0$ and $a_{4i}^1 = a_{2i}^0$ for every $1 \le i \le m$ such that $f_{p_1,\dots,p_{2m}}(y_1, \dots, y_{2m}) \notin \mathbb{Z}$ whenever $p_1, \dots, p_{2m} \in \{-2m, \dots, -1, 0, 1, \dots, 2m\}$, not all zero, and $(y_1, \dots, y_{2m}) \in B_1^1 \times \cdots \times B_{2m}^1$ where $B_i^1 = [a_{2i-1}^1, a_{2i}^1]$ $(1 \le i \le 2m)$. We also assume, changing the interval for smaller ones if needed, that $0 < a_{2i}^1 - a_{2i-1}^1 < 1/(2m)$. Call $K_1 = \cup_{i=1}^{2m} B_i^1$

Proceeding recursively, we build a infinite nested sequence

$$K_1 \supseteq \cdots \supseteq K_n \supseteq K_{n+1} \supseteq \cdots$$

such that, for every positive integer $n$:

- $K_n$ is the union of $2^n m$ compact connected intervals $B_i^n = [a_{2i-1}^n, a_{2i}^n]$ $(1 \le i \le 2^n m)$ with
$$a_{2i}^n - a_{2i-1}^n < 1/(2^n m) \tag{7.7}$$
and where $a_1^n < \cdots < a_{2^{n+1}m}^n$ and $a_{4i}^{n+1} = a_{2i}^n$ and $a_{4i-3}^{n+1} = a_{2i-1}^n$ for every $1 \le i \le 2^n m$;

- $F \subset K_n$;
- for every $p_1, \ldots, p_{2^n m} \in \{-2^n m, \ldots, -1, 0, 1, \ldots, 2^n m\}$, not all simultaneously zero, and every $(y_1, \ldots, y_{2^n m}) \in B_1^n \times \cdots \times B_{2^n m}^n$

$$f_{p_1,\ldots,p_{2^n m}}(y_1, \ldots, y_{2^n m}) \notin \mathbb{Z}. \tag{7.8}$$

We now take $K = \cap_{n=1}^{\infty} K_n$ and claim that $K$ is the desired Cantor set. Indeed, $K$ is compact, perfect and totally disconnected set of real numbers containing $F$. Firstly, $K$ is a compact set containing $F$ because it is an intersection of sets with that property; even more, $K$ contains the set $\{a_i^n : n \in \mathbb{N} \wedge 1 \leq i \leq 2^{n+1} m\}$. Secondly, $K$ is rationally independent. Indeed, for every finite sequences $y_1, \ldots, y_l \in K$ and $p_1, \ldots p_l \in \mathbb{Z}$ we can take a large enough integer $n$ such that $|p_j| \leq 2^n m$ for all $0 \leq j \leq l$ and apply (7.8) to guarantee that $\sum_{j=1}^{m} p_j y_j \notin \mathbb{Z}$ unless $p_1 = \cdots = p_l = 0$. Thirdly, $K$ is totally disconnected because, using (7.7), for any two different points $x, y \in K$ there exist some big enough $n$ and two different $i, i'$ such that $x \in B_i^n$ and $y \in B_{i'}^n$. And, finally, $K$ is perfect. Indeed, let $x \in K$ and $\varepsilon > 0$. Let $n$ be a large enough integer so $2^n m > 1/\varepsilon$, then $x \in B_i^n$ for some $1 \leq i \leq 2^n m$ and therefore $a_{2i-1}^n$ and $a_{2i}^n$ are points in $K \cap (x - \varepsilon, x + \varepsilon)$. $\square$

We are now ready to present the proof of Theorem H. Let $S$ be a nonorientable (respectively orientable) noncompact surface of finite genus $g \geq 4$ (respectively $g \geq 1$). According to Corollary 1.34, given any compact nonorientable surface $S'$ of genus $g \geq 4$ (respectively any compact orientable surface of genus $g \geq 1$), there exists a (metric compact) totally disconnected subset $K \subset S'$ such that $S' \setminus K$ is homeomorphic to $S$ (and therefore analytic diffeomorphic, see Remark 1.17).

Let $n = g$ (respectively $n = g - 1$), take $T$ a $(n, n-2)$-i.e.t. (respectively a $(2n+2)$-i.e.t.) as in Theorem 7.12 (respectively as in Remark 7.13) and let $T^c$ be the associated $(n-1, n-2)$-c.e.t. (respectively $(2n+1)$-c.e.t.). Let $M_T$ and $X_T$ be, respectively, the suspended compact surface and the vector field associated with $T$. Call $p_0$ be the only singular point of $X_T$ (the restriction of $X_T$ to $S_T = M_T \setminus \{p_0\}$ is analytic).

On account of Theorem 7.14, we are done with the proof if we are able to find a Cantor set in $S_{T^c}$, $\mathcal{K}$, containing $p_0$ and with the extra property that any other orbit of $X_T$ meeting $\mathcal{K}$ is dense backward and forwardly and meets $\mathcal{K}$ in exactly one point.

Let $k = n$ (respectively $k = 2n + 2$) and $a_1 = 0 < a_2 < \cdots < a_{k+1} = 1$ be the discontinuity points of $T$. Call, for every $1 \leq i \leq k$, $\lambda_i = a_{i+1} - a_i$. Scaling the interval $[0, 1]$ by an appropriate irrational number if necessary, there is no loss of generality in assuming that all the $\lambda_i$ are irrational numbers.

Let us consider a maximal rationally independent set $F = \{\lambda_{j_1}, \ldots, \lambda_{j_N}\}$ contained in $\{\lambda_1, \ldots, \lambda_n\}$ (i.e. any $\lambda_i \notin F$ makes $F \cup \{\lambda_i\}$ be rationally dependent). The previous observation guarantees $F$ is nonempty. We may also assume that $\lambda_{j_1} = \lambda_1$.

Use Lemma 7.15 to take a rationally independent Cantor set, $K \subset [0,1]$, containing $F$. We show that, for any $k \in \mathbb{Z}\backslash\{0\}$, $T^k(K) \cap K \neq \emptyset$. Indeed, assume by contradiction the existence of $x, y \in K$ and $k \geq 1$ such that $T^k(x) = y$. Using Equation (7.2) and the maximality of $F$, we deduce that there are some $n_x, n_y, n_1, \dots, n_N \in \mathbb{Z}$, not all vanishing, such that $n_x x + n_y y + n_1\lambda_{j_1} + \cdots + n_N\lambda_{j_N} \in \mathbb{Z}$ contradicting the rational independence of $K$. An analogous reasoning justifies that $\{\lambda_1\} = K \cap \bigcup_{i=1}^n \mathcal{O}_T(a_i)$. If we then identify again 0 and 1, $K$ can be seen as Cantor set in $\mathbb{S}^1$. Finally, call $\mathcal{K} = [K \times \{1\}] \subset S_{T^c}$ and observe that $p_0 = [(\lambda_1, 1)] \in \mathcal{K}$ and that any other orbit of $X_T$ meeting $\mathcal{K}$ does it exactly once and it is dense backward and forwardly. This completes the proof of Theorem H.

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